Solution 1

1. (Demand functions) Students 1, 2 and 3 are interviewed about their coffee consumption. More specifically, they report their *reservation prices* for the quantity of cups of coffee they would be willing to purchase. The reservation price is the highest price per quantity such that they are willing to buy that specific quantity.

Student 1:

Quantity	1	2	3
Price in £	3.00	2.00	1.00

Student 2:

Quantity	1	2	3
Price in £	4.00	1.50	0.50

Student 3:

Quantity	1	2	3
Price in £	1	0.00	0.00

a) Calculate the respective individual demand functions D_i, $i \in \{1, 2, 3\}$ (as functions in the price) and the inverse demand functions P_i, $i \in \{1, 2, 3\}$ (as functions in the quantity demanded). If you properly consider the functions as maps $Q_i\colon A \to B$ and $P_i\colon C \to E$, what sets A, B, C, E are most appropriate?

Solution: The inverse demand function associates each quantity with the corresponding reservation price. We are working under the assumption that prices, although being reported discretely in pounds and pennies, are continuous quantities. Therefore, we consider the inverse demand functions as maps $\mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$. Then we obtain

n = quantity demanded

$$
\begin{aligned}
&P_1(1) = 3, \quad P_1(2) = 2, \quad P_1(3) = 1, \quad P_1(n) = 0 \quad \forall n \geq 4; \\
&P_2(1) = 4, \quad P_2(2) = 1.5, \quad P_2(3) = 0.5, \quad P_2(n) = 0 \quad \forall n \geq 4; \\
&P_3(1) = 1, \quad P_3(n) = 0 \quad \forall n \geq 2.
\end{aligned}
$$

(student 1)
(student 2)
(student 3)

One might also wonder whether to include a quantity of 0 into the considerations. However, the corresponding reservation price then could not be uniquely determined so one deliberately does not specify this value.

$$\mathbb{1}_A = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$$

On the other hand, demand functions can be regarded as maps $\mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}$. With this perception, the demand functions take the form of step functions:

$$\begin{aligned}
D_1(p) &= \mathbb{1}_{[3,0)}(p) + \mathbb{1}_{[2,0)}(p) + \mathbb{1}_{[1,0)}(p) \quad p > 0; \\
D_2(p) &= \mathbb{1}_{[4,0)}(p) + \mathbb{1}_{[1.5,0)}(p) + \mathbb{1}_{[0.5,0)}(p) \quad p > 0; \\
D_3(p) &= \mathbb{1}_{[1,0)}(p) \quad p > 0.
\end{aligned}$$

One might again wonder what to do with the price level 0. But again, the quantity of demand for price 0 is not uniquely specified (or otherwise one could consider it to be infinity).

b) Calculate the aggregate demand function D and the corresponding inverse P of the aggregate demand function.

Solution: To calculate the aggregate demand function, we just need to add the individual demand functions. That means

$$\begin{aligned}
D(p) &= D_1(p) + D_2(p) + D_3(p) \\
&= \mathbb{1}_{[4,0)}(p) + \mathbb{1}_{[3,0)}(p) + \mathbb{1}_{[2,0)}(p) + \mathbb{1}_{[1.5,0)}(p) + 2\mathbb{1}_{[1,0)}(p) + \mathbb{1}_{[0.5,0)}(p).
\end{aligned}$$

In stark contrast to the aggregate demand function, the inverse of the aggregate demand function cannot be computed as the sum of the inverses of the individual demand functions. Instead, we really have to compute the (generalised) inverse of the aggregate demand function.[1] That is

$$\begin{aligned}
&P(1) = 4, \quad P(2) = 3, \quad P(3) = 2, \quad P(4) = 1.5, \\
&P(5) = P(6) = 1, \quad P(7) = 0.5, \quad P(n) = 0 \quad \forall n \geq 8.
\end{aligned}$$

c) If the price for one cup of coffee is £0.75, how many cups will be sold in total?

Solution: We calculate $D(0.75) = 6$. Indeed, at a price of £0.75, Student 1 will buy 3 cups, Student 2 will demand 2 cups and Student 3 will demand 1 cup.

2. (Price elasticity of demand and supply)

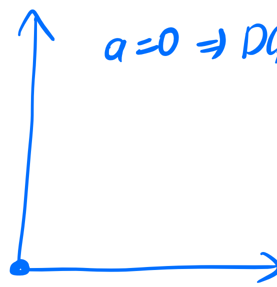
a) Student A claims that since a linear demand function has constant slope, it also exhibits constant price elasticity. ~~Is Student A correct~~? Justify your answer.

Under which conditions is student A correct?

[1] That is, $P(q) = \max\{p \geq 0 \,|\, D(p) \geq q\}$.

Solution: Student A is actually correct, but the only situation where this can happen is when we have perfect inelasticity. Let's assume some demand function D is linear. That means,

$a = 0 \Rightarrow D(p) = -bp$



$$D(p) = a - bp, \quad a \geq 0, b \geq 0, \quad 0 \leq p \leq a/b.$$

If we then calculate the price elasticity of demand, we obtain

$$\epsilon_D(p) = D'(p)p/D(p) = \frac{-bp}{a - bp}.$$

This expression is constant in p if and only if $a = 0$ or $b = 0$. If $a = 0$, the only value b can take is also $b = 0$. That means, in any case the demand function will be completely inelastic.

→ other than linear

b) In case you agree with Student A, are there other possible demand functions that exhibit a constant price elasticity? If you disagree with Student A, are there alternatives to linear demand functions that exhibit a constant price elasticity?

Solution: Assume that $\epsilon_D(p) = -c$ for $c \geq 0$ for any p. Then the demand function satisfies the following differential equation

$$D'(p) = -cD(p)/p. \Rightarrow \frac{D'(p)}{D(p)} = -\frac{c}{p} \Rightarrow$$

$(\log(D(p)))' = (-c\log p)' \Rightarrow$

Any non-negative solution to that equation is of the form

$$D(p) = a\exp(-c\log(p)) = a\,p^{-c}, \qquad a \geq 0.$$

$\log D(p) = -c\log p + k \Rightarrow$
$D(p) = \exp\{-c\log p + k\}$
$= \exp\{-c\log p\}\exp\{k\}$
$= a\exp\{-c\log p\},$
$a \geq 0$

So there are in fact more possible demand functions with constant elasticity.

c) Now, Student B claims that since a linear supply function has constant slope, it also exhibits constant price elasticity. ~~Is Student B correct~~? Justify your answer. Under which conditions is student B correct?

Solution: Student B is correct and in this case, the resulting supply function will also be interesting. Similarly to above, the supply function takes the form

$$S(p) = a + bp, \quad a \in \mathbb{R}, b \geq 0, \quad p \geq \max\{0, -a/b\}.$$

Then,

$$\epsilon_S(p) = \frac{bp}{a + bp}.$$

Again, this is constant in p if and only if $a = 0$ or $b = 0$. However, the case of $a = 0$ (with arbitrary non-negative slope b) is definitely a more flexible and interesting case.

d) In case you agree with Student B, are there other possible supply functions that exhibit a constant price elasticity? If you disagree with Student B, are there alternatives to linear supply functions that exhibit a constant price elasticity?

Solution: Due to the law of supply, the price elasticity of supply is always non-negative. If we assume that $\epsilon_S(p) = c \geq 0$ for all p, then we obtain the differential equation

$$S'(p) = cS(p)/p$$

with the solutions

$$S(p) = a\exp(c\log(p)) = a\,p^c, \qquad a \geq 0.$$

3. (Quasi-Concavity) We have seen in the lecture that one of the assumptions of a production function is quasi-concavity. If $D \subseteq \mathbb{R}^n$ is a convex subset of $\mathbb{R}^n$, we say that a function $f\colon D \to \mathbb{R}$ is quasi-concave if

$$f((1-\lambda)y + \lambda x) \geq \min\{f(x), f(y)\} \qquad \forall x, y \in D,\ \forall \lambda \in [0,1]. \qquad (1)$$

a) Show that for an interval $I \subseteq \mathbb{R}$ and $f\colon I \to \mathbb{R}$ is quasi-concave if and only if

(i) $\nearrow$ $f\uparrow$ $f(x) \leq f(\lambda x + (1-\lambda)y) \leq f(y) \Rightarrow$

$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\} = f(x)$

(i) f is monotonically increasing; or

(ii) f is monotonically decreasing; or

(iii) f is monotonically increasing and then monotonically decreasing.

Solution: If f satisfies (i) or (ii), it is obviously quasi-concave. Now, if f satisfies (iii), there is some global maximum $z \in I$. If $x \leq y \leq z$ (1) follows as in (i), if $z \leq x \leq y$, (1) follows as in (ii). If $x \leq z \leq y$, then for all $w \in [x,z]$ $f(w) \geq f(x)$ and for all $v \in [z,y]$ $f(v) \geq f(y)$ such that (1) follows.

Now assume that f satisfies not (i) and not (ii) and not (iii). Then there exist there points $x \leq z \leq y$ in D such that $f(x) > f(z)$ and $f(y) > f(z)$. This contradicts (1). $\square$

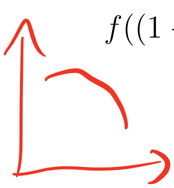
<u>concave</u>

① Def: $f((1-\lambda)y + \lambda x)) \geq (1-\lambda)f(x) + \lambda f(y)$

b) Let $D \subseteq \mathbb{R}^n$ be convex. Show that if a function $f\colon D \to \mathbb{R}$ is concave, it is also quasi-concave. Show that the reverse implication does not hold by giving a counterexample.

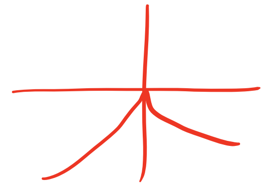
Solution: Let f be concave. That means for any $x, y \in D$ and for any $\lambda \in [0,1]$

② If f convex, $-f$ concave



$$\begin{aligned} f((1-\lambda)y + \lambda x) &\geq (1-\lambda)f(y) + \lambda f(x) \\ &\geq (1-\lambda)\min\{f(x), f(y)\} + \lambda \min\{f(x), f(y)\} \\ &= \min\{f(x), f(y)\}. \end{aligned}$$

③ $f''(x) \leq 0$ concave



Using **a)** it is not too difficult to come up with a counterexample. For example, let $D = \mathbb{R}$ and $f(x) = -|x|^{1/2}$.

c) Let $D \subseteq \mathbb{R}^n$ be convex and $f\colon D \to \mathbb{R}$ a continuously differentiable function. Show that the following assertions are equivalent:

(i) f is quasi-concave.

(ii) For all $y \in \mathbb{R}$ the set $f^{-1}([y,\infty))$ is convex.

(iii) For all $x, y \in D$: If $f(y) \geq f(x)$, then $\nabla f(x)^\top (y - x) \geq 0$.

Remark: Can you give similar (equivalent) conditions for concavity?

Proof: (i) $\Longrightarrow$ (ii): Let $x, z \in D$, $y \in \mathbb{R}$, and $x, z \in f^{-1}([y,\infty))$. Then $f(x) \geq y$ and $f(z) \geq y$. So $\min\{f(x), f(z)\} \geq y$. Using the quasi-concavity of f we obtain for $\lambda \in [0,1]$

$$f((1-\lambda)x + \lambda z) \geq \min\{f(x), f(z)\} \geq y. \quad \Rightarrow f((1-\lambda)x+\lambda z) \in [y,\infty)$$

By the definition of the pre-image, $(1-\lambda)x + \lambda z \in f^{-1}([y,\infty))$. $\Rightarrow f^{-1}(\ )$ convex

(ii) $\Longrightarrow$ (iii): Let $x, y \in D$ with $f(y) \geq f(x)$. Then, $y, x \in f^{-1}([f(x),\infty))$ and therefore also $(1-\lambda)y + \lambda x \in f^{-1}([f(x),\infty))$ for $\lambda \in [0,1]$. That means that the function $F\colon [0,1] \to \mathbb{R}$, $F(\lambda) = f((1-\lambda)y + \lambda x) - f(x)$ is non-negative and $F(1) = 0$. By a continuity argument, we can see that $F'(1) \leq 0$.[2] We obtain

$$F'(\lambda) = \nabla f((1-\lambda)y + \lambda x)^\top (x - y).$$

In summary, $F'(1) = \nabla f(x)^\top (x - y) \leq 0$, which is equivalent to what we want to show.

(iii) $\Longrightarrow$ (i): Let $x, y \in D$ and w.l.o.g. $f(y) \geq f(x)$. Assume that (i) fails to hold. Then there is some $\lambda_0 \in (0,1)$ such that $F(\lambda_0) := f((1-\lambda_0)y + \lambda_0 x) < f(x)$. But that means that there is some $\lambda' \in [\lambda_0, 1]$ such that $F(\lambda') < f(y)$ and $F'(\lambda') > 0$. However,

$$F'(\lambda') = \nabla f(x')^\top (x - y),$$

where $x' = (1-\lambda')y + \lambda' x$. We can see that $x' - y = \lambda'(x - y)$. Hence,

$$\nabla f(x')^\top (x' - y) > 0,$$

but $F(\lambda') = f(x') < f(y)$. This contradicts (iii). $\square$

[2] This argument works as follows: Since f is continuously differentiable and F is just a concatination of continuously differentiable functions, also F is continuously differentiable. Now, assume that $F'(1) > 0$. Then, due to the continuity of F', there is some $\delta > 0$ such that $F'(1-\epsilon) > 0$ for all $\epsilon \in [0,\delta]$. But that means that F is strictly increasing on $[1-\delta, 1]$. So $F(\delta) < F(1) = 0$. However, this is a contradiction such that we can derive that $F'(1) \leq 0$.

d) We say a function $g\colon D \to \mathbb{R}$ is quasi-convex if $f = -g$ is quasi-concave. State the equivalence in **c)** directly in terms of g.

Solution:

(i)

$$g((1-\lambda)y + \lambda x) \leq \max\{g(x), g(y)\} \qquad \forall x, y \in D,\ \forall \lambda \in [0,1]. \quad (2)$$

(ii) For all $y \in \mathbb{R}$ the set $g^{-1}((-\infty, y])$ is convex.

(iii) For all $x, y \in D$: If $g(y) \leq g(x)$, then $\nabla g(x)^\top (y - x) \leq 0$.

e) Prove that the following production functions are increasing and quasi-concave.

- Leontief production function $f(x_1, x_2) = \min(ax_1, bx_2)$; $\to (x_1, x_2) \leq (x_1', x_2')$ $\Rightarrow \min(ax_1, bx_2) \leq \min(ax_1', bx_2')$
- linear production function $f(x_1, x_2) = ax_1 + bx_2$;
- Cobb-Douglas production function $f(x_1, x_2) = Ax_1^a x_2^b$,

where $A, a, b, x_1, x_2 \geq 0$.

Solution:

The fact that the three production functions are increasing (or better – non decreasing) is a straight forward calculation.

For the quasi-concavity, let $(x_1, x_2), (x_1', x_2') \in \mathbb{R}^2_{\geq 0}$ and $\lambda \in [0,1]$. Then

- $$\begin{aligned} &\min\big(a(1-\lambda)x_1 + a\lambda x_1', b(1-\lambda)x_2 + b\lambda x_2'\big) \\ \geq &\min\big(a\min(x_1, x_1'), b\min(x_2, x_2')\big) \\ = &\min\big(\min(ax_1, bx_2), \min(ax_1', bx_2')\big)\,. \end{aligned}$$

- $$a(1-\lambda)x_1 + \lambda x_1' + b(1-\lambda)x_2 + b\lambda x_2' \geq \min(ax_1 + bx_2, ax_1' + bx_2')\,.$$

- To establish the quasi-concavity of a Cobb-Douglas production function is a bit more tricky. First of all, observe that the case $A = 0$ is trivial and the case $\min(a, b) = 0$ is easy, since then the Cobb-Douglas is then an increasing function of one variable only. So let $A, a, b > 0$.
  We will use point (ii) in **c)**. Let $y \geq 0$ Since we have already established that a Cobb-Douglas function is increasing, we obtain that

  $$f^{-1}([y, \infty)) = f^{-1}(\{y\}) + \mathbb{R}^n_{\geq 0} = \{(x_1, x_2) + (z_1, z_2) \mid f(x_1, x_2) = y \text{ and } z_1, z_2 \geq 0\}\,.$$

That means it is sufficient to show that the graph describing the contour line / isoquant $f^{-1}(\{y\})$ belongs to a convex function. Let $g_y \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$g_y(x_1) = x_2 \quad \Longleftrightarrow \quad f(x_1, x_2) = y\,.$$ $\Leftrightarrow A x_1^a x_2^b = y \Leftrightarrow$

This yields that $g_y(x_1) = C x_1^{-a/b}$ where $C = y^{1/b} A^{-1/b} \geq 0$. The second derivative is given by

$A x_1^a (g_y(x_1))^b = y \Leftrightarrow$

$\vdots$

$$g_y''(x_1) = C \frac{a}{b} \left(\frac{a}{b} - 1 \right) x_1^{a/b-2} \geq 0\,.$$

$g_y(x_1) = y^{1/b} A^{-1/b} x_1^{-a/b}$

This shows that g_y is a convex function which proves the claim.

Remark: Of course, one can also use one of the above equivalent conditions for quasi-concavity. However, you should pay attention with the Leontief production function since it is not differentiable.