Problem Sheet 2

We will discuss the solutions of this problem sheet in the problem class on Friday, 3 February 2023.

1. Decide if the following statements are true or false. Explain and justify your answers.

a) Every monotone and quasi-concave production function exhibits increasing, decreasing or constant returns to scale.

b) The quasi-concavity of a production function implies that if we mix certain bundles of inputs we will always be able to produce not less than with any of the single bundles.

2. Consider a production function $f\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{>0}$

$$f(x_1, x_2) = \frac{2}{1 + \frac{1}{x_1 x_2}}\,.$$

a) Show that f is a homothetic function.

b) Show that f is non-decreasing and quasi-concave.

c) Calculate the elasticity of scale of f. For which $(x_1, x_2) \in \mathbb{R}^2_{\geq 0}$ exhibits f locally increasing, decreasing or constant returns to scale.

d) Calculate the MRTS of f and show that it is positively homogeneous of degree 0.

e) Show that any differentiable homothetic production function has an MRTS which is homogeneous of degree 0.

3. Let $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ be a non-decreasing and quasi-concave production function. Show that following statements.

a) The factor demand function $\underline{x}^*\colon \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is positively homogeneous of degree 0.

b) The profit function $\pi^*\colon \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is positively homogeneous of degree 1.

c) The profit function π^* is non-decreasing in $\underline{p} \in \mathbb{R}^m_{\geq 0}$ and non-increasing in $\underline{w} \in \mathbb{R}^n_{\geq 0}$.

d) The profit function π^* is convex.

4. **(Envelope Theorem)** The Envelope Theorem asserts the following. Let $\varphi\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, be some continuously differentiable function with partial derivatives $\partial_1\varphi$, $\partial_2\varphi$. Define the function $\Phi\colon \mathbb{R} \to \mathbb{R}$

$$\Phi(a) = \max_{x \in \mathbb{R}} \varphi(x, a).$$

Assume that Φ is well defined and differentiable. Let $x^*\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$x^*(a) = \arg\max_{x \in \mathbb{R}} \varphi(x, a),$$

where we assume that the argmax is unique and x^* is differentiable and takes only values in the interior of D. Then

$$\Phi'(a) = \partial_2\varphi(x^*(a), a).$$

a) Prove the Envelope Theorem.

b) Give an argument how one can use the Envelope Theorem to derive Hotelling's Lemma.