Solution 3

1. Suppose you observe a firm using two input goods with prices (w_1, w_2) and one output good with price p. You have

t	p	w_1	w_2	x_1	x_2	y
1	2	2	1	10	20	100
2	3	1	2	14	10	110

WAPM:
$p^1y^1 - w_1^1x_1^1 - w_2^1x_2^1 \geq$
$p^1y^2 - w_1^1x_1^2 - w_2^1x_2^2$ ✓

Check whether the Weak Axiom of Profit Maximisation is satisfied and check whether the Weak Axiom of Cost Minimisation is satisfied.

Solution: We first check the WAPM. At time point $t = 1$ the firm makes a profit of:

$$p^1y^1 - w_1^1x_1^1 - w_2^1x_2^1 = 200 - 40 = 160\,.$$

Let's check what they would have obtained if they had used the output and input at time $s = 2$ with the prices at time $t = 1$:

$$p^1y^2 - w_1^1x_1^2 - w_2^1x_2^2 = 220 - 38 = 182 > 160\,.$$

So we see that the WAPM is violated and we don't need to check the time $t = 2$.

Now, we consider the WACM. Here, we actually only need to consider one constellation. At time point $t = 1$ we have costs:

$$w_1^1x_1^1 + w_2^1x_2^1 = 40\,.$$

However, if they had used the input combination from time $s = 2$, they would have had a cost of only:

$$w_1^1x_1^2 + w_2^1x_2^2 = 38$$

and they could have produced more, since $y^2 = 110 > 100 = y^1$.

2. Suppose the two input goods have prices $w_1 > 0$ and $w_2 > 0$. In the lecture we have seen that a necessary first-order condition for the cost minimisation problem for some fixed output $y > 0$ is given by

$$\mathrm{MRTS}(x_1^*, x_2^*) = -\frac{w_1}{w_2} \tag{1}$$

for the cost minimising input bundle $\underline{x}^* = (x_1^*, x_2^*) \in \mathbb{R}^2_{\geq 0}$ if the production function is differentiable. Graphically, the general situation can be illustrated as in figure 1 for a firm with increasing, continuous and quasi-concave production function $f\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$.

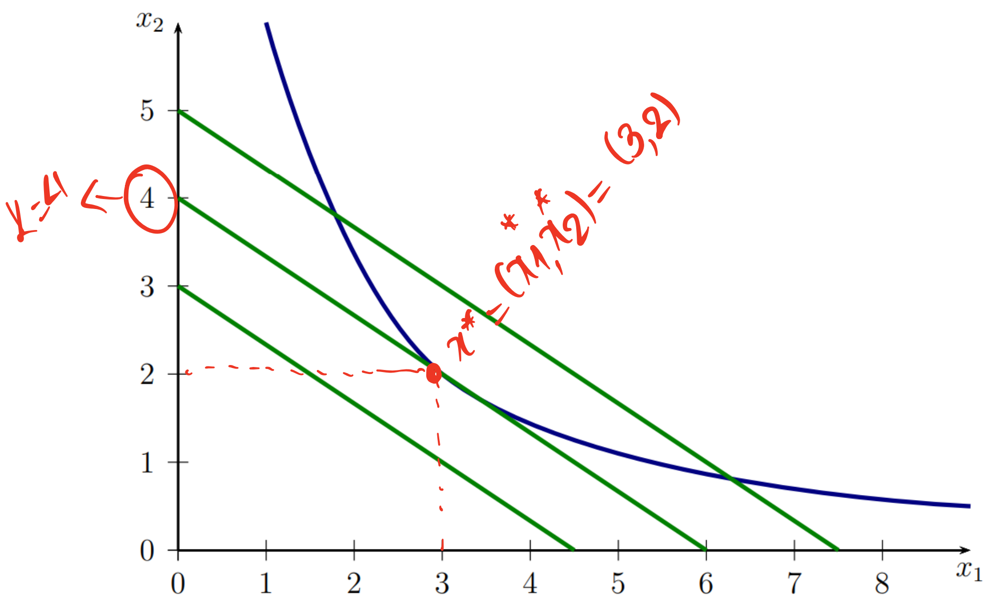


Figure 1: Several lines (green) with slope $-w_1/w_2$ and different intercepts. The curve (blue) is the isoquant $f^{-1}(\{y\})$.

a) Why is the cost minimising consumption bundle $\underline{x}^*$ necessarily on the isoquant $f^{-1}(\{y\})$?

Solution: The firm needs to produce at least y. That is, necessarily $\underline{x}^* \in f^{-1}([y, \infty))$. Assume that $\underline{x}^* \in f^{-1}((y, \infty))$. Since (y, ∞) is open and f is continuous, also $f^{-1}((y, \infty))$ is open. But that means that if $\underline{x}^* \in f^{-1}((y, \infty))$ then there is another $\underline{x}' \in f^{-1}((y, \infty))$ with $\underline{x}' < \underline{x}^*$ (componentwise). Since the prices are strictly positive, this implies that $\underline{x}'$ had lower costs than $\underline{x}^*$. Hence $\underline{x}^* \in f^{-1}(\{y\})$.

b) What does a line with slope $-w_1/w_2$ and intercept $K \geq 0$ represent in this context economically?

Solution: For positive prices, a line with slope $-w_1/w_2$ and intersection $K \geq 0$ in the $x_1 - x_2-$plane can be represented as the set of input bundles (x_1, x_2) satisfying:

$$x_2 = K - \frac{w_1}{w_2}x_1.$$

This is equivalent to:

$$w_1 x_1 + w_2 x_2 = c,$$

where $c = Kw_2$. That means a line with slope $-w_1/w_2$ and intercept $K \geq 0$ represents possible input bundles with the same total cost $c = Kw_2$ at prices w_1 and w_2. If you wish you could call them 'isocost lines'.

c) Determine the value of $-w_1/w_2$ using Figure 1.

Solution: We can see that the slope of the lines is $-\frac{4}{6} = -\frac{2}{3}$. Thus

$$-\frac{w_1}{w_2} = -\frac{2}{3}.$$

d) What does condition (1) mean graphically?

Solution: It means we have costs such that the associated isocost line is tangential to the isoquant.

e) Determine the cost minimising input bundle $\underline{x}^* = (x_1^*, x_2^*)$ in figure 1. What can you say about the total costs when using $\underline{x}^*$?

Solution: We can directly see in the graphic that $\underline{x}^* = (x_1^*, x_2^*) = (3, 2)$.

Actually, we cannot determine the total cost. As seen in the solution of part **b)** the total costs are $c^* = K^* w_2$ where $K^* = 4$ is the intercept of the isocost line to which $\underline{x}^*$ belongs.

f) Suppose that the price for good 1 increases to $w_1' > w_1$ whereas the price for good 2 remains constant. Determine the new cost minimising input bundle graphically using figure 1.

Solution: Figure 2 depicts the situation. The new isocost line, drawn in red, has a 'smaller' slope (in absolute terms). We see that the new cost minimising input bundle $\underline{x}'^*$ will be such that $x_1'^* < x_1^*$ and $x_2'^* > x_2^*$.

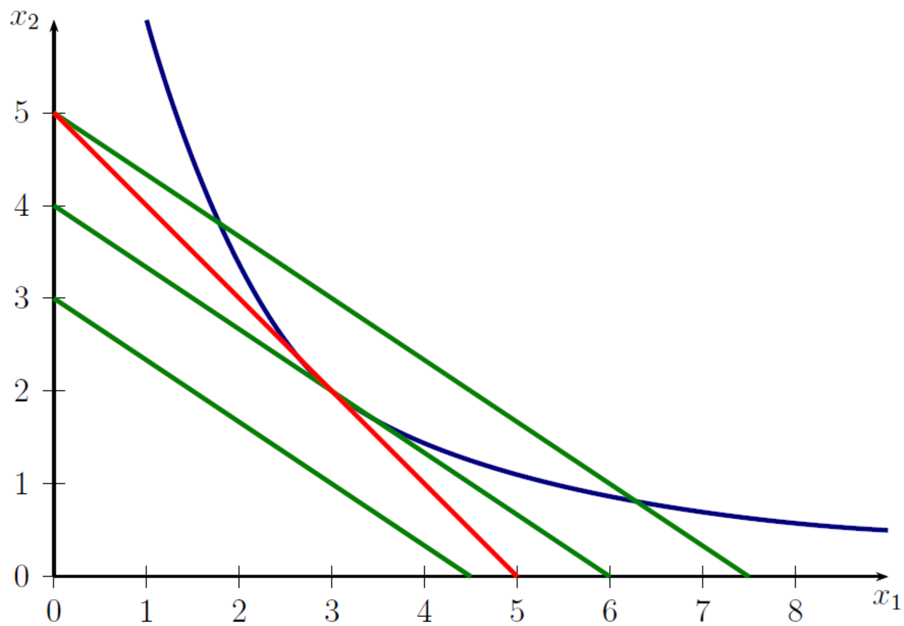


Figure 2: The red line is the new isocostline.

g) Now consider a Leontief production function $f(x_1, x_2) = \min(x_1, x_2)$. Determine the cost minimising input bundle graphically, similarly to the approach in figure 1. To this end, consider the following situations.

- $y = 2$ and $w_1 = w_2 = 1$.
- $y = 2$, $w_1 = 2$ and $w_2 = 1$.

Use the same graph. (Label the axes adequately.)

Solution: We can see the situation in figure 3. The slope of the isocost curve changes. However, since the isoquant of a Leontief production function is not differentiable, the optimal input bundle remains the same and is always $(x_1^*, x_2^*) = (y, y) = (2, 2)$.

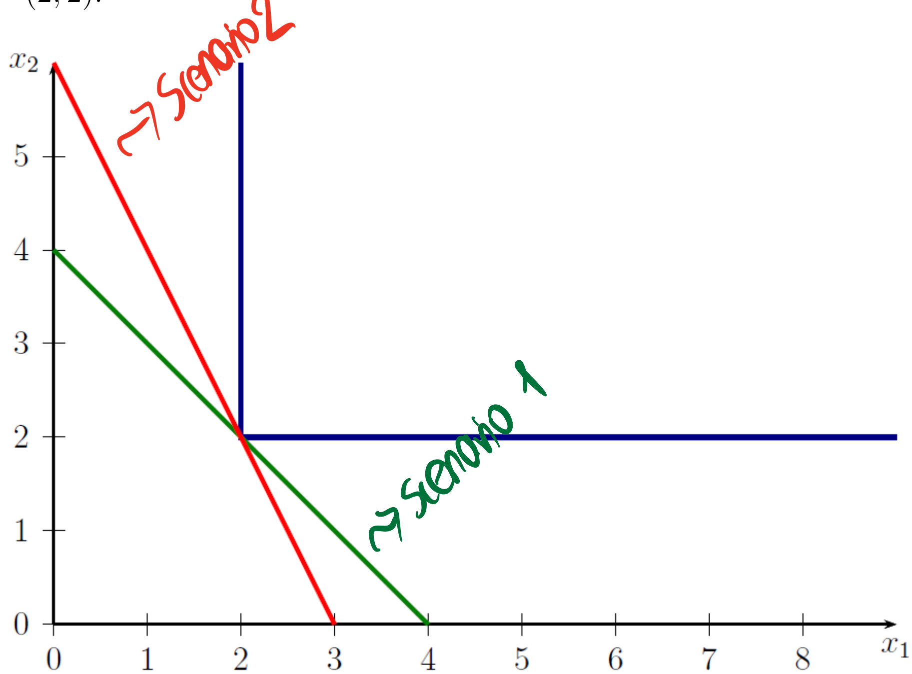


Figure 3: The green line is the isocost curve with costs $w_1 = w_2 = 1$ and the red line is the new isocostline with $w_1 = 2$ and $w_2 = 1$.

3. Consider a firm with three inputs. The first two inputs are used for the actual production with a Cobb-Douglas technology whereas the third one limits the maximal amount of output. Moreover, assume that the first two input goods are variable in the short run, but the third one is fixed in the short run and only variable in the long run.Therefore, the production function takes the form:

$$f\colon \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}, \qquad f(x_1, x_2, x_3) = \min\{x_1^{1/3} x_2^{2/3}, x_3\}$$

a) Think of an example where such a production function might occur (also with the fixed and variable costs).

Solution: An example might be a firm producing with the input goods x_1, x_2 with a Cobb-Douglas technology. However, the firm has to pay a rent x_3 for its factory (or its machines) and the amount of machines or the size of the factory limits the maximal production.

In this example, it is also plausible that the size of the factory or the quantity of machines cannot be adapted so easily. So it is plausible that x_3 is fixed in the short run, but variable in the long run.

b) Does f satisfy the two conditions we imposed on production functions (monotonicity and quasi-concavity)? What is the behaviour of f with respect to scale?

Solution: M: $\underline{x} \leq \underline{z} \Rightarrow f(\underline{x}) \leq f(\underline{z})$

Let $\underline{x}, \underline{z} \in \mathbb{R}^3_{\geq 0}$ such that $\underline{x} \leq \underline{z}$ (that means $x_1 \leq z_1$, $x_2 \leq z_2$ and $x_3 \leq z_3$). There are many ways to show the claim. One is to invoke a telescope argument:

$$\begin{aligned} f(z_1, z_2, z_3) - f(x_1, x_2, x_3) &= f(z_1, z_2, z_3) - f(x_1, z_2, z_3) \\ &\quad + f(x_1, z_2, z_3) - f(x_1, x_2, z_3) \\ &\quad + f(x_1, x_2, z_3) - f(x_1, x_2, x_3) \geq 0. \end{aligned}$$

Each summand is non-negative such that it easily follows that the left-hand side of the equation is also non-negative.

To show quasi-concavity, you can directly verify the definition. Alternatively, you can use the fact that a Cobb-Douglas production function is quasi-concave. Hence, for $\underline{x}, \underline{z} \in \mathbb{R}^3_{\geq 0}$, $\lambda \in [0, 1]$ and $\underline{x}' = (1 - \lambda)\underline{x} + \lambda \underline{z}$, we obtain

$$x_1'^{1/3} x_2'^{2/3} \geq \min\{x_1^{1/3} x_2^{2/3}, z_1^{1/3} z_2^{2/3}\}$$

and clearly

$$x_3' \geq \min\{x_3, z_3\}.$$

x_3 x_3' z_3

Combining this yields

$$f(\underline{x}') = \min\{x_1'^{1/3} x_2'^{2/3}, x_3'\} \geq \min\{x_1^{1/3} x_2^{2/3}, z_1^{1/3} z_2^{2/3}, x_3, z_3\} = \min\{f(\underline{x}), f(\underline{z})\}.$$

We can directly see that f is positively homogeneous of degree 1. So f has constant returns to scale.

c) Compute the cost function and the short run cost function.

Solution: We start with the short-run cost function c_s^*. Due to the very form of the production function, there is no solution to the short-run cost function for $x_3 < y$. So that's why we assume that $x_3 \geq y$. We first calculate the short-run conditional factor demand for good 1 and 2. That is, we seek, for $y \geq 0$

$$\underset{x_1^{1/3}x_2^{2/3}=y}{\arg\min}\; w_1x_1 + w_2x_2 + w_3x_3 = \underset{x_1^{1/3}x_2^{2/3}=y}{\arg\min}\; w_1x_1 + w_2x_2$$

$(x_1, x_2 > 0)$

To obtain uniqueness, we confine attention to the case $w_1, w_2, w_3 > 0$. Then the constraint implies that $x_1 = y^3/x_2^2$. Substituting in the objective, we seek the minimiser of

$$w_1\frac{y^3}{x_2^2} + w_2x_2.$$

This gives the first-order condition

$$-2w_1\frac{y^3}{x_2^3} + w_2 = 0.$$

[SOC satisfied - no need to check!]

From that we obtain the solution for the conditional factor demand for good 2 as

$$x_2^*(w_1, w_2, w_3, y) = y\left(\frac{2w_1}{w_2}\right)^{1/3}.$$

Hence,

$$x_1^*(w_1, w_2, w_3, y) = \frac{y^3}{(x_2^*)^2} = y\left(\frac{w_2}{2w_1}\right)^{2/3}.$$

So we can compute the conditional short-run cost function as

$$\begin{aligned}
c_s^*(w_1, w_2, w_3, x_3, y) &= w_1x_1^*(w_1, w_2, w_3, y) + w_2x_2^*(w_1, w_2, w_3, y) + w_3x_3\\
&= y\left(w_1^{1/3}\left(\frac{w_2}{2}\right)^{2/3} + w_2^{2/3}\left(2w_1\right)^{1/3}\right) + w_3x_3\\
&= \theta\, y\, w_1^{1/3}w_2^{2/3} + w_3x_3,
\end{aligned}$$

where $\theta = 2^{1/3}\frac{3}{2}$.

To determine the long-run, we do not need to invoke any kind of differential calculus. We can just argue that – since prices are positive – it is a waste of money if $x_1^{1/3}x_2^{2/3} \neq x_3$. Indeed, if $x_1^{1/3}x_2^{2/3} < x_3$ we could decrease the amount of x_3, thus, save money without diminishing the output. On the other hand, if $x_1^{1/3}x_2^{2/3} > x_3$, with the same argument, we could decrease the input of x_1 and x_2 to some extent without affecting the level of output. So

$$x_3^*(w_1, w_2, w_3, y) = y.$$

On the other hand, the conditional factor demand functions for good 1 and 2 are not affected. So we have

$$c^*(w_1, w_2, w_3, y) = y\left(\theta\, w_1^{1/3}w_2^{2/3} + w_3\right).$$

d) Suppose that the prices w_1, w_2, w_3 are such that

$$c^*(y) = 8y, \qquad\qquad c_s^*(x_3, y) = 6y + 2x_3,$$

where c_s^* is only defined for $x_3 \geq y$. Can you explain why this is the case?

Solution: Indeed, we can see that the cost functions take that form. As already explained above, there is no possibility to produce y if $x_3 < y$. Moreover, we can see that:

$$c^*(y) \leq c_s^*(x_3, y)$$

if and only if $x_3 \geq y$.

e) Using the cost functions from **d)** describe the optimal short-run and long-run behaviour of the firm, if the output price is $p = 5$, $p = 7$, and $p = 10$, and if the fixed input is initially at $x_3 = 10$.

Solution: The short-run profit function takes the form

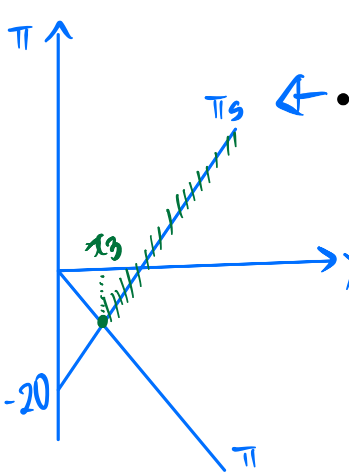
$$\pi_s^*(y) = py - c_s^*(y) = y(p - 6) - 20.$$

The long-run profit function takes the form

$$\pi^*(y) = py - c^*(y) = y(p - 8).$$

p	π_s	π
$p=5$	$-y-20$	$-3y$
$p=7$	$y-20$	$-y$
$p=10$	$4y-20$	$2y$

- $p = 5:$ Both the short-run and the long-run profit functions are affine functions with a negative slope. So the best thing to do is to shut down the production, both in the short run (profit of -20) and in the long run (0 profit).



- $p = 7:$ The short-run profit function is affine with positive slope. So the best thing to do is to produce at the maximal short-run capacities, which is, to produce an output of $x_3 = 10$. But still, we are only able to have a negative profit of -10. In the long run, we have a linear function with negative slope. So quite interestingly, even though the short-run solution is to increase output at its maximum, the best thing to do is to shut down the factory in the long run. Then we have 0 profits.

- $p = 10$: In the short run, we have an affine function with positive slope. So the best thing to do is to produce with maximal capacities, that is, to produce $y = x_3 = 10$. Then, we will have a positive profit of 20. In the long run, we have a linear function with positive slope. So the best thing to do is to produce an infinite amount which would yield an infinite profit.

f) Now, determine the optimal short-run and long-run behaviour for the firm if the prices are given by the inverse demand function $p(y) = 10 - y$ for $0 \le y \le 10$ (again, assuming that $x_3 = 10$ initially).

Solution: The short-run profit function has the form

$$\pi_s^*(y) = p(y)y - c_s^*(y) = (10 - y)y - 6y - 20 = -y^2 + 4y - 20.$$

It is maximised at $y = 2$ with a profit of -16.

The long-run profit function has the form

$$\pi^*(y) = p(y)y - c^*(y) = (10 - y)y - 8y = -y^2 + 2y.$$

It has its maximum at $y = 1$ where we obtain a profit of 1.