Problem Sheet 4

We will discuss the solutions of this problem sheet in the problem class on Friday, 3 March 2023.

1. Consider the lexicographic order $\preceq$ on $\mathbb{R}^2$. That means for any $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\underline{y} = (y_1, y_2) \in \mathbb{R}^2$ it holds that $\underline{x} \preceq \underline{y}$ if and only if:

$$(x_1 < y_1) \qquad \text{or} \qquad (x_1 = y_1 \quad \text{and} \quad x_2 \leq y_2).$$

a) Check which properties of preferences defined in the lecture (completeness, transitivity, continuity, strong monotonicity, local nonsatiation, strict convexity) the lexicographic order satisfies.

b) Show that if there is a utility function $u \colon \mathbb{R}^2 \to \mathbb{R}$ representing $\preceq$, u is an injection.

2. Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a convex and closed set. Let $u \colon X \to \mathbb{R}$ be a continuous, strictly monotone, strictly quasi-concave utility function.

a) Show that for any k such that there is some $x \in X$ with $u(\underline{x}) = k$ the expenditure function $e(\cdot, k) \colon \mathbb{R}^n_{\geq 0} \to [0, \infty)$ is concave (so it is concave in the prices).

b) Let $n = 2$, $u(x_1, x_2) = x_1^a x_2^b$ with $a, b > 0$. Calculate the indirect utility function v, expenditure function e, Marshallian demand x^* and Hicksian demand x^*_H.

c) Verify that the expenditure function you obtain in **(b)**, as a function in the prices (so for fixed utility level) is nondecreasing, homogeneous of degree 1 and concave.

d) Now suppose you have an alternative representation of the ordinal utility which is induced by u given by $u_{\log} \colon X \to \mathbb{R}$, $u_{\log}(x_1, x_2) = \log(u(x_1, x_2))$. Compute the associated quantities: indirect utility function $v_{\log}$, expenditure function $e_{\log}$, Marshallian demand $x^*_{\log}$ and Hicksian demand $x^*_{\log,H}$.

3. Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a convex and closed set. Let $u\colon X \to \mathbb{R}$ be a continuous, strictly monotone, strictly quasi-concave utility function.

a) Let $v\colon \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the indirect utility function.

- Prove that for any $\underline{p} > 0$ the function $v(\underline{p}, \cdot)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is strictly increasing.
- Prove that for any $m \geq 0$ the function $v(\cdot, m)\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ quasi-convex. Recall that a function f is quasi-convex if $-f$ is quasi-concave; see question 3 on Problem Sheet 1.

b) Assume that the prices for the goods are strictly positive, $\underline{p} > 0$, and income is positive, $m > 0$. Is it possible that all goods are inferior? Prove your claim.