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1. [default,M2]

   In the framework of the N-period binomial model with constant parameters $S_0 = 8, u = 2, d = 1/2, r = 0$, let $S = (S_n)_{n=0}^N$ be the stock price process, M_n its historical *minimum* up to time n (i.e. $M_n := \min_{i=0,\dots,n} S_i$). Consider the down-and-in rebate option with the lower barrier $L = 6$ which expires at time N and pays 1 if S_n is less than L for any $n = 0, \dots, N$; in other words, this derivative has a payoff $1 - Y_N$ at maturity N, where $Y_n := 1_{\{M_n \geq 6\}}$ (i.e. $Y_n = 1$ if $M_n \geq 6$, and $Y_n = 0$ otherwise). We denote with V_n the arbitrage-free price at time $n = 0, \dots, N$ of this option.

   Below, whenever we say that a process is Markov, we mean with respect to the risk-neutral measure $\mathbb{Q}$ and with the usual filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n), n = 0, \dots, N$ generated by the coin tosses $X_n(\omega) = \omega_n$ on the probability space $\Omega = \{H, T\}^N$. Answer the following questions and justify carefully with either proofs or counterexamples.

   (a) Draw the binary tree representing S. Can you draw it as a recombinant tree?

   A. No **B. Yes**

   (b) Are $(X_n)_{n \leq N}$ independent under $\mathbb{Q}$?

   A. No **B. Yes**

   (c) Are $(X_n)_{n \leq N}$ identically distributed under $\mathbb{Q}$?

   A. No **B. Yes**

   (d) Compute $\mathbb{Q}(\{\omega\})$ for every $\omega \in \{H, T\}^N$, then choose the correct statement

   A. $\mathbb{Q}(\{\omega\})$ is constant in $\omega \in \{H, T\}^N$

   B. $\mathbb{Q}(\{\omega\})$ is not constant, but depends only on the number of heads in $\omega \in \{H, T\}^N$

   C. None of the above

   (e) Is S Markov?

   A. No **B. Yes**

   (f) Is M a Markov? **A. No** B. Yes

   (g) Is Y a Markov? **A. No** B. Yes

   (h) Is (S, M) a Markov process? A. No **B. Yes**

   (i) Is (S, Y) a Markov process? A. No **B. Yes**

   (j) Is (S, Y, M) a Markov process? A. No **B. Yes**

   (k) Which of the above processes $W = (W_n)_n$ are Markov and are such that, for every $n = 0, \dots, N$, V_n admits the representation $V_n = v_n(W_n)$, for some Borel $v_n : \mathbb{R} \to \mathbb{R}$?

   A. S B. M C. Y **D.** (S, M) **E.** (S, Y) **F.** (S, Y, M)

   (l) Among several the choices of W which you selected in the previous question, which one would be best (i.e. lead to the shortest computations) to use to compute price explicitly

   A. S B. M C. Y D. (S, M) **E.** (S, Y) F. (S, Y, M)

Solution:

(a) S has constant $U = 2, D = 1/2$ and so it is permutation invariant, i.e. it is represented by a recombining tree

(b) The corresponding risk neutral probabilities are

$$\tilde{P}_n = \frac{1 + r - D}{U - D} = \frac{1 + 0 - 1/2}{2 - 1/2} = \frac{1}{3}, \quad \tilde{Q}_n = 1 - \tilde{P}_n = \frac{2}{3}.$$

Thus, in this particular case $\tilde{P}_n$ is actually deterministic (i.e. constant in ω), which shows that the coin tosses are independent under $\mathbb{Q}$, since

$$\tilde{P}_n(\omega_1, \ldots, \omega_n) := \mathbb{Q}(X_{n+1} = H | (X_1, \ldots, X_n) = (\omega_1, \ldots, \omega_n)). \tag{1}$$

(c) Since

$$\mathbb{Q}(X_{n+1} = H | X(n) = \omega(n)) = \tilde{P}_n(\omega(n)) = \frac{1}{3}$$

is deterministic, if equals $\mathbb{Q}(X_{n+1} = H)$. Thus $\mathbb{Q}(X_{n+1} = H) = 1/3$ does not depend on n, so the coin tosses are identically distributed

(d) Since the coin tosses are IID and $\mathbb{Q}(X_{n+1} = H) = 1/3$, we have $\mathbb{Q}((X(n)) = (\omega(n))) = \frac{1}{3^{H_n}}(\frac{2}{3})^{n-H_n}$, where $H_n = H_n(\omega(n))$ is the number of Heads.

(e) As usual, to show that S is $\mathbb{Q}$-Markov, we try to write S_{n+1} as a function of S_n (which is $\mathcal{F}_n$-measurable, since it only depends on the first n coin tosses, i.e. it is a function of $(X_1, \ldots, X_n)$), and a rv B_n which is independent (under $\mathbb{Q}$) from $\mathcal{F}_n$, and then apply the independence lemma to get that $\mathbb{E}^{\mathbb{Q}}[f(S_{n+1})|\mathcal{F}_n]$ equals $g(S_n)$ for some g. As we often do, we take $B_n := S_{n+1}/S_n$. Since X_{n+1} is independent of $\mathcal{F}_n$, the identity $S_{n+1}/S_n = h(X_{n+1})$ (where h is the function $h(H) = U$, $h(T) = D$) shows that $B_n = S_{n+1}/S_n$ is independent on $\mathcal{F}_n$. Thus, it follows from the independence lemma that

$$\mathbb{E}^{\mathbb{Q}}[f(S_{n+1})|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[f\left(S_n h(X_{n+1})\right)|\mathcal{F}_n] = g(S_n), \tag{2}$$

where g is the function

$$g(s) := \mathbb{E}^{\mathbb{Q}}[f\left(sh(X_{n+1})\right)] = \mathbb{Q}(X_{n+1} = H)f(su) + \mathbb{Q}(X_{n+1} = T)f(sd) = \frac{1}{3}f(su) + \frac{2}{3}f(sd).$$

Remark: Notice that we don't actually need to compute g explicitly to conclude that S is Markov. However, for applications (for example to pricing) one should compute g as explicitly as possible, so you might as well do that here.

(f) M is not Markov. Indeed after drawing its tree we immediately see the problem: $M_2(HH) = 8 = M_2(HT)$ yet $M_3(HTT) = 4$ is different from $M_3(HHH) = M_3(HHT) = M_3(HTH) = 8$, so

$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(HH) = \tilde{p}f(8) + (1 - \tilde{p})f(8), \quad \text{does not equal}$$
$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(HT) = \tilde{p}f(8) + (1 - \tilde{p})f(4)$$

whenever $f(8) \neq f(4)$, where $\tilde{p} = \mathbb{Q}(H) = 1/3$.

(g) Y is not Markov. Indeed after drawing its tree we immediately see the problem: $Y_2(HH) = 1 = Y_2(HT)$ yet $Y_3(HTT) = 0$ is different from $Y_3(HHH) = Y_3(HHT) = Y_3(HTH) = 1$, so

$$\mathbb{E}_2^{\mathbb{Q}} f(Y_3)(HT) = \tilde{p}f(1) + (1-\tilde{p})f(1), \quad \text{does not equal}$$
$$\mathbb{E}_2^{\mathbb{Q}} f(Y_3)(TT) = \tilde{p}f(1) + (1-\tilde{p})f(0)$$

whenever $f(1) \neq f(0)$, where $\tilde{p} = \mathbb{Q}(H) = 1/3$.

(h) (S, M) is Markov, as it is easily guessed after drawing its tree. To prove it, define

$$C_{n+1} := \frac{S_{n+1}}{S_n} = \begin{cases} u, & \text{if } X_{n+1} = H \\ d, & \text{if } X_{n+1} = T \end{cases},$$

so that $M_{n+1} = M_n \wedge (S_n C_{n+1})$. Then by the independence lemma

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1}, M_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n C_{n+1}, M_n \wedge (S_n C_{n+1}))] = g(S_n, M_n),$$

where

$$g(s,m) := \mathbb{E}^{\mathbb{Q}}[f(sC_{n+1}, m \wedge (sC_{n+1}))].$$

(i) (S, Y) is Markov, as it is easily guessed after drawing its tree. To prove it, define Define $h(x) := 1_{\{x \geq 6\}}, x \in \mathbb{R}$, notice that $h(x \wedge y) = h(x)h(y)$, so that $Y_n = h(M_n)$ and so

$$Y_{n+1} = h(M_{n+1}) = h(M_n \wedge (S_n C_{n+1})) = Y_n h(S_n C_{n+1}).$$

Then by the independence lemma

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1}, Y_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n C_{n+1}, Y_n h(S_n C_{n+1}))] = z(S_n, Y_n),$$

where

$$z(s,y) := \mathbb{E}^{\mathbb{Q}}[f(sC_{n+1}, yh(sC_{n+1}))].$$

(j) (S, Y, M) is Markov. Indeed by the independence lemma

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1}, Y_{n+1}, M_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n C_{n+1}, Y_n h(S_n C_{n+1}), M_n \wedge (S_n C_{n+1}))] = v(S_n, Y_n, M_n),$$

where

$$v(s,y,m) := \mathbb{E}^{\mathbb{Q}}[f(sC_{n+1}, yh(sC_{n+1}), m \wedge (sC_{n+1}))].$$

Alternatively, one could more simply write

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1}, Y_{n+1}, M_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n C_{n+1}, h(M_n \wedge S_n C_{n+1}), M_n \wedge (S_n C_{n+1}))] = w(S_n, M_n),$$

where

$$w(s,m) := \mathbb{E}^{\mathbb{Q}}[f(sC_{n+1}, h(m \wedge sC_{n+1}), m \wedge (sC_{n+1}))].$$

(k) One could choose X to be (S, M), or (S, Y), or (S, Y, M); for each such process, one could also write *explicitly* v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \ldots, N-1$ (instead of just proving the existence of v_n), which is necessary to compute prices (though you were not required to do this to answer the question).

Let us do that for the process (S, Y), and then for (S, M). The risk neutral pricing formula gives

$$V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}]$$

Since

$$v_n(S_n, Y_n) = V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}] = \frac{1}{1+r}\mathbb{E}_n^{\mathbb{Q}}[v_{n+1}(S_{n+1}, Y_{n+1})] = z(S_n, Y_n),$$

where

$$z(s, y) := \mathbb{E}^{\mathbb{Q}}[f(sC_{n+1}, yh(sC_{n+1}))], \qquad h(x) := 1_{\{x \geq 6\}},$$

we get that

$$v_n(s, y) = \frac{1}{3}v_{n+1}(su, y1_{\{su \geq 6\}}) + \frac{2}{3}v_{n+1}(sd, y1_{\{sd \geq 6\}}), \quad 0 \leq n \leq N-1,$$

and of course $v_N(s, y) = 1 - y$. As for (S, M): write

$$v_n(S_n, M_n) = V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}] = \frac{1}{1+r}\mathbb{E}_n^{\mathbb{Q}}[v_{n+1}(S_{n+1}, M_{n+1})] = g(S_n, M_n),$$

where

$$g(s, m) := \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[v_{n+1}(sC_{n+1}, m \wedge (sC_{n+1}))],$$

we get that

$$v_n(s, m) = \frac{1}{3}v_{n+1}(su, m \wedge (su)) + \frac{2}{3}v_{n+1}(sd, m \wedge (sd)), \quad 0 \leq n \leq N-1,$$

and of course $v_N(s, m) = 1 - 1_{\{m \geq 6\}} = 1_{\{m < 6\}}$.

(l) One should choose (S, Y), because it takes fewer values than (S, M) and than (S, Y, M)