This document contains 2 questions.

1. [default,Q6]

An *exchange option* with maturity $T \geq 0$ gives its holder the right (but not the obligation!) to receive one unit of S^1 and give in return one unit of S^2 at time T, where S^1 and S^2 are stocks in two different companies. Suppose today a hedge-fund is willing to trade with you an exchange option with expiration in 3 months for 2 and the prices of the stocks are $S_0^1 = 100, S_0^2 = 97$. Can you make an arbitrage?

A. Yes: buy the option, etc. B. Yes: sell the option, etc. C. No D. Not enough info to answer

In the answers, by 'etc' I mean that you can additionally trade in S^1, S^2, and the bank account.

Solution: The exchange option's payoff is $(S_T^1 - S_T^2)^+$. To understand the solution, it helps to consider first what happens in the degenerate case in which the expiry T is at time 0. In this case, quite clearly you should buy the option and one share of S^2, and short-sell one share of S^1: this way you never risk losing money and you are left with $-2 - 97 + 100 = 1$, so this is an arbitrage.

If $T > 0$ the only difference is that you should put your £1 in the bank and wait until T. This is an arbitrage, since at time $T = 3$ months the payoff X_T of this strategy will be:

1. If $S_T^2 < S_T^1$ then you should exercise the option, and obtain a payoff

$$X_T = \underbrace{(S_T^1 - S_T^2)}_{option} + \underbrace{(S_T^2 - S_T^1)}_{stocks} + \underbrace{(1 + r)}_{bank} = 1 + r > 0.$$

2. If $S_T^2 \geq S_T^1$ then the option expires worthless and so $X_T = (S_T^2 - S_T^1) + (1 + r) > 0$

2. [default,O13]

In a market composed of one underlying, and of a bank account with interest rate r between now and time T, consider the following derivatives:

1. Option F gives its buyer the right, and the obligation, to buy at expiry $T > 0$ the underlying at a price $K \in \mathbb{R}$ (fixed in advance), instead of at its market price P_T (so, the buyer of the option will have to buy the underlying at a price K, under all possible circumstances). Thus, the final value (payoff) of option F is $F_T = P_T - K$. Notice that, if the value of K is set equal to the T-forward price $P_0(1+r)$, where P_0 is the initial price of the underlying, then option F is the forward contract and its initial value (price) is $F_0 = 0$.
2. Option A *(resp. B)* gives its buyer the right, but not the obligation, to buy *(resp. to sell)*, at expiry T, the underlying at a price K (so, at expiry the buyer can choose whether buy the underlying at a price K, or not). Fyi, option A is called a *call* option, and option B a *put* option.
3. Option C *(resp. D)* gives its buyer the obligation, but not the right, to buy *(resp. to sell)*, at expiry T, the underlying at a price K (so, at expiry the seller of the option can choose whether its buyer should buy the underlying at a price K, or not, and the buyer will be obligated to do what the seller decided).

Notice that the value of $K \in \mathbb{R}$ (which is known as the *strike price*) is arbitrary, and so the initial prices of these options are in general not 0. Call X_t the value of option (X) (for $X \in \{A, B, C, D\}$) at time $t \in \{0, T\}$. We will use the following standard notation: by definition, the positive (*negative*) part of $x \in \mathbb{R}$ is $x^+ := \max(x, 0)$ and $x^- := -\min(x, 0)$ (the minus sign in from the *min* is not a typo!). Notice that $x^- = (-x)^+$ and $x = x^+ - x^-$ for all $x \in \mathbb{R}$.

(a) Write the payoffs of the above options as functions of the underlying:

i. What is the payoff of option A ?
A. $P_T - K$ B. $(K - P_T)^+$ **C.** $(P_T - K)^+$ D. $-(P_T - K)^-$

ii. What is the payoff of option B ?
A. $P_T - K$ **B.** $(K - P_T)^+$ C. $(P_T - K)^+$ D. $-(K - P_T)^-$

iii. What is the payoff of option C ?
A. $K - P_T$ B. $(K - P_T)^+$ **C.** $-(P_T - K)^-$ D. $-(K - P_T)^-$

iv. What is the payoff of option D ?
A. $(P_T - K)^+$ B. $(K - P_T)^+$ C. $-(P_T - K)^-$ **D.** $-(K - P_T)^-$

(b) For what value of $(a, b) \in \mathbb{R}^2$ do the payoffs of options A, B, F satisfy the linear relation $aA_T + bB_T = F_T$?
A. $(1, -1)$ B. $(1, 1)$ C. $(-1, 1)$ D. $(1, 0)$

(c) If K equals the T-forward price of the underlying, and the put option has price B_0 at time 0, what is the price A_0 of the call option?
A. 0 **B.** B_0 C. $B_0 + P_0$ D. $B_0/(1 + r)$

(d) Using your intuition, can you tell which of the prices A_0, B_0, C_0, D_0 should always be positive (i.e. ≥ 0), and which should always be negative (i.e. ≤ 0)?
A. $A_0 \geq 0$, $B_0 \leq 0$, $C_0 \geq 0$, $D_0 \leq 0$
B. $A_0 \geq 0$, $B_0 \geq 0$, $C_0 \geq 0$, $D_0 \geq 0$
C. $A_0 \leq 0$, $B_0 \leq 0$, $C_0 \geq 0$, $D_0 \geq 0$

D. $A_0 \geq 0$**,** $B_0 \geq 0$**,** $C_0 \leq 0$**,** $D_0 \leq 0$

(e) Consider the *strangle* option, whose payoff equals that of a put option with strike price K_1 plus that of a call option with strike price $K_2 > K_1$ (both options have the same underlying S and expiry T). For what function f is the strangle's payoff $= f(S_T)$?

A.
$$f(x) = \begin{cases} x - K_2 & \text{if } x > K_2 \\ 2x - K_1 - K_2 & \text{if } K_1 \leq x \leq K_2 \\ K_1 - x & \text{if } x < K_1 \end{cases}$$

B. $f(x) = K_2 - K_1$

C.
$$f(x) = \begin{cases} x - K_1 & \text{if } x > K_2 \\ 0 & \text{if } K_1 \leq x \leq K_2 \\ K_2 - x & \text{if } x < K_1 \end{cases}$$

D.
$$f(x) = \begin{cases} x - K_2 & \textbf{if } x > K_2 \\ 0 & \textbf{if } K_1 \leq x \leq K_2 \\ K_1 - x & \textbf{if } x < K_1 \end{cases}$$

(f) Complete the following sentence:
If $K_1 < P_0 < K_2$ and I want to buy the above portfolio, it means that I believe that the underlying's price P_T will

A. change quite dramatically in either direction.

B. decrease quite dramatically.

C. increase quite dramatically.

D. not change much.

E. not increase much.

F. not decrease much.

Solution: We will use the (standard) notations $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$ to denote the positive and negative parts of $x \in \mathbb{R}$.

(a) Fyi, A is called a *call* option, and B a *put* option.

(b) Since $x^- = (-x)^+$, the identity $x = x^+ - x^- = x^+ - (-x)^+$ applied to $x = P_T - K$ gives $A_T - B_T = (P_T - K)^+ - (K - P_T)^+ = P_T - K = F_T$.

(c) By the law of one price $A_T - B_T = F_T$ implies $A_0 - B_0 = F_0$, and so $A_0 = B_0$ since $F_0 = 0$.

(d) Whatever option gives a right but no obligation must have a positive value, and whatever option gives an obligation but no rights must have a negative value.

(e) Combining the payoffs from item (a) we get the answer. Fyi, this derivative is called a *strangle*.

(f) Only if the underlying will change price enough to exit the interval $[K_1, K_2]$ will the payoff be positive