This document contains 3 questions.

1. [default,Q8]

   Consider the one-period binomial model where the bank account has interest rate $r = 0$, and the stock takes values $S_0 := 4$, and S_1 can take the values 5 and 3.

   (a) Is this model free of arbitrage?

   A. No **B. Yes**

   (b) What should be the price P_0 for a put option on S with strike $K = 4$?

   A. 0 **B. 1/2** C. 1 D. None of the above

   Suppose that such a put option was sold at a price 1. To realise an arbitrage, how many shares $h \in \mathbb{R}$ should I buy/sell if

   (c) I *sell* one option?

   A. No arbitrage is possible B. $h = 0$ C. $h = -1$ **D.** $h \in [-1, 0]$ E. None of the above

   (d) I *buy* one option?

   A. No arbitrage is possible B. $h = 0$ C. $h = -1$ D. $h \in [-1, 0]$ E. None of the above

Solution:

(a) Yes, because $\frac{3}{4} = d < 1 + r = 1 < u = \frac{5}{4}$.

(b) By replication we find that the put option should be sold at, since this is the initial capital of a replicating portfolio. This is found by solving the equation $V_1^{x,h} = P_1$, where

$$V_1^{x,h} := x(1 + r) + h(S_1 - S_0),$$

is the final wealth of the portfolio (x, h) with initial wealth x and h shares of the stock S, and $P_1 = (K - S_1)^+$ is the payoff of the put option. Indeed, since $r = 0$, $V_1^{x,h} = P_1$ becomes

$$\begin{cases} x + h(5-4) = (4-5)^+ \\ x + h(3-4) = (4-3)^+ \end{cases}, \quad \text{i.e.} \begin{cases} x + h = 0 \\ x - h = 1. \end{cases}$$

whose unique solution is $x = 1/2$, $h = -1/2$.

The final wealth which I obtain if starting with initial wealth x and buying h shares of the stock S and g put options is

$$V_1^{x,h,g} := x(1 + r) + h(S_1 - S_0) + g(P_1 - P_0).$$

By definition, the portfolio (x, h, g) is an arbitrage if $x = 0$, $V_1^{x,h,g} \geq 0$ and $V_1^{x,h,g} \neq 0$. So, to find an arbitrage we have to solve the system $V_1^{0,h,g} \geq 0$ and find out if is has any non-zero solutions.

(c) Selling one option means $g = -1$; in this case the system $V_1^{0,h,g} \geq 0$ becomes

$$\begin{cases} h - (0-1) \geq 0 \\ -h - (1-1) \geq 0 \end{cases},$$

which has solution $-1 \leq h \leq 0$. If $h \in (-1, 0)$, then both inequalities hold strictly. If $h = -1$ or $h = 0$ then one inequality holds with equality, but the other holds strictly. So, any $h \in [-1, 0]$ is an arbitrage.

(d) Buying one option means $g = 1$; in this case the system $V_1^{0,h,g} \geq 0$ becomes

$$\begin{cases} h + (0-1) \geq 0 \\ -h + (1-1) \geq 0 \end{cases},$$

which has no solution: no arbitrage is possible. This is obvious: at $p = 1 > 1/2$ the option is overpriced, so one should sell it, not buy it!

2. [default,O9]

Consider a trinomial market model. This is the model consisting of one bond with risk-free interest rate $r > -1$, and one stock with price $S_0 > 0$ at time 0, and whose price at time 1 it takes the three values d, m, u with probability respectively $q, 1-(p+q), p$; we assume that $0 < d < m < u$ and $p, q \in (0, 1)$. For some values of parameters (in the range described above) this model free of arbitrage, and for some it is not; on which of the parameters p, q, d, m, u, r does this depend on?

A. all of them B. d, m, u, r **C.** d, u, r D. p, q, d, u, r E. None of the above

Solution: Note that $1 - (p+q) = 0$ iff S does not take the value m, in which case this actually becomes the binomial model for which we already know that the NA condition is $d < 1 + r < u$.

Let us how that, also in the trinomial model, there is no arbitrage iff $d < 1 + r < u$; in particular, the value of m does not matter (as long as $d < m < u$). Notice that the proof is identical to the one for the binomial model (seen in class), and applies also to any model *with only one underlying* S (plus the bank account) which only takes finitely many values: in this case u and d should be the largest and the smallest value taken by S_1/S_0.

W.l.o.g. we take as probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and assume that $S_1(\omega_1) = uS_0, S_1(\omega_2) = mS_0, S_1(\omega_3) = dS_0$. The final value V_1 of the portfolio with initial capital V_0 and trading strategy H is $V_1 = HS_1 + (1+r)(X_0 - HS_0)$. An arbitrage is a portfolio with 0 initial capital and whose final value V_1 satisfies $V_1 \geq 0$ a.s. and $\mathbb{P}(V_1 > 0) > 0$. Thus, H is an arbitrage iff $V_1(\omega) = H(S_1(\omega) - (1+r)S_0)$ is ≥ 0 for all $\omega \in \Omega$ and > 0 for at least one $\omega \in \Omega$. Clearly if $H = 0$ then $V_1 = 0$ and so such H is never an arbitrage. Since $S_1(\omega_1) > S_1(\omega_2) > S_1(\omega_3)$, if $H > 0$ then $V_1(\omega_1) > V_1(\omega_2) > V_1(\omega_3)$, and so such H is an arbitrage iff $0 \leq V_1(\omega_3) = H(dS_0 - (1+r)S_0)$, i.e. iff $d \geq 1 + r$. Analogously if $H < 0$ then $V_1(\omega_1) < V_1(\omega_2) < V_1(\omega_3)$, and so such H is an arbitrage iff $0 \leq V_1(\omega_1) = H(uS_0 - (1+r)S_0)$, i.e. iff $u \leq 1 + r$. In summary, there is an arbitrage iff either $d \geq 1 + r$ or $u \leq 1 + r$, proving the thesis.

3. [default,O3c]

Define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	4	10
$Y_1(\omega)$	4	2	-4

Consider a one-period trinomial model (B, S) with interest rate $r = 1$ (so $B_0 = 1, B_1 = 1 + r$), and a stock whose price is given by $S_0 = 2$ at time 0 and by S_1 at time 1. Here $\Omega = \{x_1, x_2, x_3\}$ is the underlying probability space, on which is defined a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$).

(a) Is this model free of arbitrage?

A. No **B. Yes**

(b) Consider the derivative with payoff Y_1 at time 1. Is Y_1 replicable (in the model (B, S))?

A. No **B. Yes**

(c) Is the model (B, S) *complete*, i.e., can *any* option be replicated in this model?

A. No B. Yes

(d) What is the smallest price p at which an infinitely risk-averse agent would be willing to sell Y_1?

A. 1 B. 2 C. There exists no such p D. Not enough info to answer E. None of the above

Solution:

(a) **1st Solution:** With the same proof that applies for the binomial model, it is easy to show the trinomial model is free of arbitrage iff $d < 1 + r < u$. Since in this exercise the down, middle and up factors d, m, u are respectively $1, 2, 5$, we get that $1 + r = 2$ satisfies $d < 1 + r < u$, so the model is free of arbitrage.

2nd Solution: It is enough to compute the set $\mathcal{M}$ of equivalent martingale measures and show that it is not empty. Recall that $\mathbb{Q} \in \mathcal{M}$ if $S_0 = \mathbb{E}^{\mathbb{Q}}[S_1/(1 + r)]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{x_i\})$ satisfy

$$\begin{cases} 2 = q_1 + 2q_2 + 5q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

Subtracting twice the second line from the first line we get $0 = -q_1 + 3q_3$ and so $q_1 = 3q_3$ and the second line now gives $q_2 = 1 - q_1 - q_3 = 1 - 4q_3$. Imposing $q_i > 0$ we obtain that the set of q_i's corresponding to $\mathcal{M}$ is

$$\left\{ q_t := \begin{pmatrix} 3t \\ 1 - 4t \\ t \end{pmatrix} : t \in \left(0, \frac{1}{4}\right) \right\}, \tag{EMM}$$

which is non-empty.

(b) **1st Solution:** We can solve this by computing explicitly the solution to the replication equation. The portfolio with initial wealth x and trading strategy h has payoff $V_1 = (1+r)x + h(S_1 - S_0(1+r))$ equal to

$$2x \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2h \cdot \begin{pmatrix} 1-2 \\ 2-2 \\ 5-2 \end{pmatrix} = 2 \begin{pmatrix} x-h \\ x \\ x+3h \end{pmatrix}.$$

Solving for $Y_1 = V_1$ gives

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x-h \\ x \\ x+3h \end{pmatrix}$$

which has solution $x = 1, h = -1$, so Y_1 is replicable (starting with initial wealth 1 and short-selling 1 stock, and depositing the remaining $x - h \cdot S_0 = 1 - (-1) \cdot 2 = 3$ in the bank).

2nd Solution: It is enough to show that $\mathbb{E}^{\mathbb{Q}}[Y_1]/(1+r)$ is constant across all $\mathbb{Q} \in \mathcal{M}$. Using (EMM) this means that $2(3t) + 1(1-4t) + (-2)t = 1$ is constant over $t \in \left(0, \frac{1}{4}\right)$, which is clearly true.

(c) **1st Solution:** The market is not complete, since the equation $X_1 = V_1$ does not have solution for arbitrary X_1, since it corresponds to a system of 3 linear equations in 2 unknowns, and thus it does not have a solution for some values of X_1. In other words, the set of vectors of the form

$$V_1^{x,h} = 2x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2h \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

for some $x, h \in \mathbb{R}$, i.e. the vector space generated by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

does not span the vector space of all possible values of derivatives (which is $\mathbb{R}^3$ in this example), because it only has dimension 2.

Notice that the above solution suggests that the answer depends on the initial value S_0 of the stock. This is not so, as it is made clear by describing the portfolio using k, h instead of x, h. Indeed in this case

$$V_1^{k,h} := kB_1 + hS_1 = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

which obviously does not depend on S_0. Of course the vector space generated by the final wealth is the same as before, so the answer does not depend on whether we use k, h or x, h to describe our portfolio.

2nd Solution: (EMM) shows that $\mathcal{M}$ is not a singleton, which implies that the market is not complete.

(d) Since Y_1 admits a replicating strategy $x' = 1, h' = -1$, the answer is $Y_0 = x' = 1$, which is the initial capital of a replicating portfolio. Indeed, by definition

$$p := \inf\{x : V_1^{x,h}(\omega) \geq Y_1(\omega) \quad \forall \omega \in \{\omega_1, \omega_2, \omega_3\}\},$$

and so $p \leq Y_0 = 1$ follows from the fact that $V_1^{1,-1} = Y_1$ does indeed (trivially) satisfy $V_1^{1,-1} \geq Y_1$.

Now suppose by contradiction that $p < Y_0 = x'$; by definition of p, there exist x, h such that $p < x < x'$ and $V_1^{x,h} \geq Y_1$. In this case $h - h'$ is an arbitrage, because (by linearity) it has a final payoff

$$V_1^{0,h-h'} = V_1^{x,h} - V_1^{x',h'} + (x' - x)(1 + r) \geq Y_1 - Y_1 + (x' - x)(1 + r) > 0.$$

Since our model has no arbitrage, we conclude that $p = x'$.