This document contains 2 questions.

1. [default,O1]

Consider a single-period financial model with interest rate $r = 1/10$, where one can trade a stock at the price $S_0 = 100$, the forward contract on the stock, and the call option on the stock with the strike $K = 110$ at the price $C_0 = \frac{70}{11}$. Assume that the forward price of the stock is $F = 110$, and that the stock price S_1 at maturity takes the following possible values: $70, 90, 110, 130$. A model is said to be *complete* if every derivative is replicable in such model. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is this model arbitrage-free?

A. No **B. Yes**

(b) Is this model complete?

A. No B. Yes

(c) What is the smallest price at which an infinitely risk-averse investor would sell the put option with strike $K = 110$?

A. $\frac{80}{11}$ **B. $\frac{70}{11}$** C. None of the above

Solution:

(a,b) 1^{st} **solution:** As always, the forward contract can be replicated by buying one share of the underlying, and borrowing the required cash S_0 from the bank. Thus the market made of bond, stock, forward and call has the same set of possible final wealths as the market made only of bond, stock, and call. This means that the final wealth is a linear function of three variables; since in this model the underlying probability space has 4 points, this means that the replication equation has 4 equalities and only 3 variables, and so it does not always have a solution, i.e. the market is incomplete.

Let us show that the market admits no arbitrage. If $h := (h^1, h^2, h^3)$ represents the number of shares, forwards, and calls bought, by definition the portfolio (x, h) is an arbitrage iff $V_1^{(0,h)}(\omega) \geq 0$ for all ω, and > 0 holds for at least one ω. Thus, we look for a solution h of the system

$$0 \leq V_1^{(0,h)} = h^1(S_1 - (1+r)S_0) + h^2(F_1 - (1+r)F_0) + h^3(C_1 - (1+r)C_0)$$

i.e. of

$$\begin{cases} (70-110)h^1 - 40h^2 + (0-7)h^3 \geq 0 \\ (90-110)h^1 - 20h^2 + (0-7)h^3 \geq 0 \\ (110-110)h^1 \qquad\quad + (0-7)h^3 \geq 0 \\ (130-110)h^1 + 20h^2 + (20-7)h^3 \geq 0 \end{cases} \tag{1}$$

and see if there is any which does not satisfy all inequalities with equality. To eliminate h^3 and ease the calculations we define $s^1 := -20h^1, s^2 := -20h^2, s^3 := 7h^3$ and write

$$\begin{cases} 2s^1 + 2s^2 \geq s^3 \\ s^1 + s^2 \geq s^3 \\ 0 \geq s^3 \\ \frac{7}{13}s^1 + \frac{7}{13}s^2 \leq s^3 \end{cases}$$

Since s^1, s^2 only appear in the combination $s^1 + s^2$, this suggests changing variable from s^1 to $y := s^1 + s^2$, which gives the following system in the variables y, s^2, s^3

$$\begin{cases} 2y \geq s^3 \\ y \geq s^3 \\ 0 \geq s^3 \\ \frac{7}{13}y \leq s^3 \end{cases} \tag{2}$$

which can of course be considered also as a system in the variables y, s^3. As such, we just need to eliminate s^3 to get to the system

$$\begin{cases} 2y \geq \frac{7}{13}y \\ y \geq \frac{7}{13}y \\ 0 \geq \frac{7}{13}y \end{cases}$$

whose only solution is $y = 0$. Thus the set of solutions of eq. (2) is

$$\{(y, s^2, s^3) \in \{0\} \times \mathbb{R} \times \{0\}\},$$

and for any solution all the inequalities in eq. (2) hold with equality. Thus, the set of solutions of eq. (1) is $\{(t, -t, 0) : t \in \mathbb{R}\}$, and for any solution all the inequalities hold with equality. So, there is no arbitrage.

2^{nd} **solution:** Since the payoffs of the relevant securities are given by

event	ω_1	ω_2	ω_3	ω_4
stock	70	90	110	130
forward	-40	-20	0	20
call	0	0	0	20

and $\mathbb{Q}$ is an EMM (equivalent martingale measure) iff $X_0(1+r) = \mathbb{E}^{\mathbb{Q}}[X_1]$ for every traded security X, $\mathbb{Q}$ is an EMM iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 110 = 70q_1 + 90q_2 + 110q_3 + 130q_4 & \text{(Stock)} \\ 0 = -40q_1 - 20q_2 + 20q_4 & \text{(Forward)} \\ 7 = 20q_4 & \text{(Call)} \\ 1 = q_1 + q_2 + q_3 + q_4 & \text{(Probability)} \\ q_1 > 0 \quad q_2 > 0 \quad q_3 > 0 \quad q_4 > 0 & (\mathbb{Q} \sim \mathbb{P}) \end{cases}$$

Observe that the third equation gives $q_4 = 7/20$, which satisfies $q_4 > 0$, and the 1^{st} equation is the sum of the 4^{th} equation multiplied by 110 and the 2^{nd} equation. Thus, dividing times 20 the 2^{nd} equation, we get that the above system is equivalent to the system

$$\begin{cases} -\frac{7}{20} &= -2q_1 & -q_2 & \\ \frac{13}{20} &= q_1 & +q_2 & +q_3 \\ & q_1 > 0 & q_2 > 0 & q_3 > 0 \end{cases}$$

which has 3 unknowns and 2 equations; thus, if it has a solution then it has infinitely many solutions. The solution of the system of two equations is

$$q_2 = -2q_1 + \frac{7}{20}, \quad q_3 = q_1 + \frac{6}{20}$$

which satisfies inequalities $q_i > 0$, $i = 1, \dots, 3$, iff $0 < q_1 < \frac{7}{40}$. Thus there are multiple EMMs, so by the fundamental theorems the model is arbitrage free, but incomplete.

(c) 1^{st} **Solution:** We should first try to replicate the put P: if it can be replicated starting with initial capital P_0, then the answer is P_0.

Instead of setting up the appropriate linear system, we can actually easily see that the put is indeed replicable, since the identity $x = x^+ - (-x)^+$ (where $x^+ := \max(x, 0)$) applied to $x = S_1 - K$ gives that the payoff of the put option satisfies

$$(K - S_1)^+ = (S_1 - K)^+ - (S_1 - F) + (K - F),$$

and so it follows that one can replicate the put option by the following portfolio of the traded securities:

- one call option,
- one short position in the forward contract,
- the amount $\frac{K-F}{1+r}$ invested into the bank account.

Of course instead of one short position in the forward, one could equivalently short one share of the stock and deposit profits S_0 in the bank.

Thus the put option has a unique arbitrage-free price, given by

$$P_0 = C_0 + 0 + \frac{K - F}{1 + r} = \frac{70}{11}$$

Fyi, the classic result discussed in this solution is called *put-call parity.*

2^{nd} **Solution:** The arbitrage-free prices of the put option are given by the values of

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[(K - S_T)^+]$$

where $\mathbb{Q}$ spans the set of EMM. Substituting the numerical values we calculated before we get that

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[(K-S_T)^+] = \frac{70}{11}$$

for any EMM $\mathbb{Q}$, so $\frac{70}{11}$ is the unique arbitrage-free arbitrage-free of the put, and so also the smallest price at which an infinitely risk-averse investor would sell the put.

2. [default,O29]

Consider the following model with two stocks and a bank account with interest rate is $r = 1$. The prices at time $t = 0$ equal to $S_0^1 = 5$ and $S_0^2 = 5$. The prices of the two stocks at time $t = 1$ are given by the following vectors:

$$S_1^1 = \begin{pmatrix} 12 \\ 12 \\ 8 \\ 6 \end{pmatrix} \text{ and } S_1^2 = \begin{pmatrix} 16 \\ 8 \\ 6 \\ 4 \end{pmatrix}.$$

Is this model free of arbitrage?

A. No **B. Yes**

Solution: To work in discounted terms, we compute:

$$\overline{S}_1^1 - \overline{S}_0^1 = \begin{pmatrix} 6 \\ 6 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -2 \end{pmatrix}, \qquad \overline{S}_1^2 - \overline{S}_0^2 = \begin{pmatrix} 8 \\ 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \\ -3 \end{pmatrix}, \qquad \overline{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and in particular the discounted value of the portfolio (x, h^1, h^2) at time $t = 1$ is

$$\overline{V}_1^{x,h} = \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} + h^1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -2 \end{pmatrix} + h^2 \begin{pmatrix} 3 \\ -1 \\ -2 \\ -3 \end{pmatrix}.$$

We write the system $\overline{V}_1^{0,h} \geq 0$, i.e.

$$\begin{cases} h^1 + 3h^2 \geq 0 \\ h^1 - h^2 \geq 0 \\ -h^1 - 2h^2 \geq 0 \\ -2h^1 - 3h^2 \geq 0 \end{cases}.$$

To minimise the number of fractions appearing, we first eliminate h^1: we write

$$\begin{cases} -3h^2 \leq h^1 \\ h^2 \leq h^1 \\ -2h^2 \geq h^1 \\ -\frac{3}{2}h^2 \geq h^1 \end{cases} \tag{3}$$

from which we get the system

$$\begin{cases} -3h^2 & \leq & -2h^2 \\ -3h^2 & \leq & -\frac{3}{2}h^2 \\ h^2 & \leq & -2h^2 \\ h^2 & \leq & -\frac{3}{2}h^2 \end{cases}$$

whose unique solution is $h^2 = 0$. To solve (3) we then take any

$$h^1 \in [\max(-3h^2, h^2), \min(-2h^2, -\frac{3}{2}h^2)] = \{0\},$$

i.e. the only solution of $\overline{V}_1^{0,h} \geq 0$ is $h = 0$, and so there is no arbitrage.