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1. [default,O3a]

On the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, on which is defined probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$, define random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	1	6	12
$X_1(\omega)$	22	30	44
$Y_1(\omega)$	22	32	44

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 and interest rate $r = 1$, a one stock whose initial price is $S_0 = 3$, and whose final price S_1 is as in the above table. Consider also the derivatives with payoffs X_1, Y_1. Denote with $u(X) := u(X; B, S)$ (resp. $d(X) := d(X; B, S)$) the smallest (resp. largest) value at which an infinitely risk-averse agent, investing in the market (B, S), is willing to sell (resp. buy) X. So far, all prices were stated in a fixed currency, say £. When solving this exercise, compute all values not in terms of £ but in terms of units of bond. In other words, given a process of prices $W = (W_0, W_1)$, consider instead the discounted process

$$\overline{W}_t := W_t/B_t, \text{ i.e. } \quad \overline{W}_0 = W_0, \quad \overline{W}_1 = W_1/(1 + r)$$

(so e.g. taking $W = B$ this means $\overline{B}_0 = \overline{B}_1 = 1$, taking $W = S$ this means $\overline{S}_0 = S_0, \overline{S}_1 = S_1/(1 + r)$). Recall that a model is called *complete* if any derivative can be replicated in such model.

In item (g), we consider the enlarged market (B, S, Y), where we are assuming that Y_1 is being sold at price $Y_0 := 16$ at time 0. From item (h) (included) onwards, we consider the enlarged market (B, S, X), where we are assuming that X_1 is being sold at price $X_0 := 31/2$ at time 0.

(a) Is the model (B, S) free of arbitrage?

A. No **B. Yes**

(b) Is the model (B, S) complete?

A. No B. Yes

(c) Is X_1 replicable in the model (B, S)?

A. No B. Yes

(d) Is Y_1 replicable in the model (B, S)?

A. No **B. Yes**

(e) What are $d(X; B, S), u(X; B, S)$?

A. 15,15 B. 16,16 **C. 15,16** D. $\frac{31}{2}, \frac{31}{2}$ E. None of the above

(f) What are $d(Y; B, S), u(Y; B, S)$?

A. 15,15 **B. 16,16** C. 15,16 D. $\frac{31}{2}, \frac{31}{2}$ E. None of the above

(g) Is the model (B, S, Y) complete?

A. No B. Yes

(h) Is the model (B, S, X) arbitrage-free?

A. No **B. Yes**

(i) Is the model (B, S, X) complete?

A. No **B. Yes**

Solution:

1. **1st solution** It is easy to show the trinomial model is free of arbitrage iff $d < 1 + r < u$, with the same proof that applies for the binomial model. Since in this exercise the down, middle and up factors d, m, u are respectively $1/3, 2, 4$ and $1 + r = 2$, the inequalities $d < 1 + r < u$ are satisfied.

   2nd solution Alternatively one can compute the set $\mathcal{M}$ of equivalent martingale measures and show that it is not empty. Recall that $\mathbb{Q} \in \mathcal{M}$ if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$ (where $\bar{W}_n := W_n/(1 + r)^n$ denotes the discounted process W), $\mathbb{Q}$ is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{x_i\})$ satisfy

$$\begin{cases} 3 = q_1/2 + 3q_2 + 6q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

   Subtracting second line from twice the first line we get $5 = 5q_2 + 11q_3$ and so $q_3 = \frac{5}{11}(1 - q_2)$ and the second line now gives

$$q_1 = 1 - q_2 - q_3 = (1 - q_2) - \frac{5}{11}(1 - q_2) = \frac{6}{11}(1 - q_2).$$

   Imposing $q_i > 0$ we obtain that the set of q_i's corresponding to $\mathcal{M}$ is

$$\left\{ q_t := \begin{pmatrix} \frac{6}{11}(1-t) \\ t \\ \frac{5}{11}(1-t) \end{pmatrix} : t \in (0, 1) \right\}, \tag{EMM}$$

   which is non-empty.

2. **1st solution** We have to determine whether the replication equation $V_1^{x,h} = P_1$ has a solution for an arbitrary payoff P_1, where our final wealth is given by

$$V_1^{x,h} := x(1 + r) + h(S_1 - S_0(1 + r)).$$

   This can be expresses in discounted terms by dividing everything times $1 + r$ to get $\overline{V}_1^{x,h} = \overline{P}_1$, where

$$\bar{V}_1^{x,h} := x + h(\bar{S}_1 - \bar{S}_0).$$

   Here x is to be interpreted as a random variable with constant value x. In other words, $\bar{V}_1^{x,h}$ is a linear combination of the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \bar{S}_1 - \bar{S}_0 = \begin{pmatrix} \frac{1}{2} - 3 \\ 3 - 3 \\ 6 - 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 0 \\ 3 \end{pmatrix} \tag{1}$$

Thus, the set of attainable (discounted) wealth (i.e. the set of all possible values of $\bar{V}_1^{x,h}$) is a vector space with dimension 2. Thus this market is not complete, since set of all possible values of derivatives is in this example $\mathbb{R}^3$, which is a vector space of dimension 3, is thus strictly bigger. In other words, the replication equation $V_1^{x,h} = P_1$ does not have solution $(x,h) \in \mathbb{R}^2$ for arbitrary payoff P_1, since it corresponds to a system of 3 linearly independent equations in 2 unknowns (which does not always have a solution).

2nd solution This model is not complete, since (EMM) shows that $\mathcal{M}$ is not a singleton.

(c,d,e,f) **1st solution** Before giving the details, let us just describe our strategy. We first try to solve the replication equation. If this has a solution $(x,h) \in \mathbb{R}^2$ then V_1 is replicable and it has a unique arbitrage-free price x; if this equation has no solution then V_1 is not replicable, and so the set of arbitrage free prices is (d,s), where s is the minimum value of x for which

$$\bar{V}_1 \leq x + h(\bar{S}_1 - \bar{S}_0)$$

for some h, and analogously d is the maximum value of x for which there exists a h such that

$$\bar{V}_1 \geq x + h(\bar{S}_1 - \bar{S}_0).$$

Of course, V_1 is replicable iff $d = u$ and in that case $d = u = x$, and so one could also solve the problem by computing directly d, u, without checking first whether X is replicable.

Let us now see the details. Using (1) we can write the replication equality for X_1 as

$$\begin{pmatrix} x \\ x \\ x \end{pmatrix} + h \begin{pmatrix} -\frac{5}{2} \\ 0 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 22 \\ 30 \\ 44 \end{pmatrix} . \tag{2}$$

The second eq. gives $x = 15$, so the third eq. gives $3h = 22 - 15$, i.e. $h = 7/3$, and now the LHS of the first eq. becomes $15 - \frac{5}{2} \cdot \frac{7}{3} = \frac{55}{6}$, which does not equal 11; thus X_1 is not replicable.

Replacing 30 with 32 in (2) gives the replication equality for Y_1; the second eq. gives $x = 16$, so the third eq. gives $3h = 22 - 16$, i.e. $h = 2$, and now the LHS of the first eq. becomes $16 - \frac{5}{2} \cdot 2 = 11$, which equals the RHS; thus Y_1 is replicable and its afp if $x = 16$.

To find the set (i, s) of afp of X_1 we consider the super-replication inequality

$$\begin{pmatrix} x \\ x \\ x \end{pmatrix} + h \begin{pmatrix} -\frac{5}{2} \\ 0 \\ 3 \end{pmatrix} \geq \frac{1}{2} \begin{pmatrix} 22 \\ 30 \\ 44 \end{pmatrix} .$$

which corresponds to the the system of inequalities

$$\begin{cases} h \leq \dfrac{2}{5}(x - 11) \\ 0 \geq 15 - x \\ h \geq \dfrac{1}{3}(22 - x)\,; \end{cases} \tag{3}$$

By definition s is the smallest $x \in \mathbb{R}$ for which there exists an $h \in \mathbb{R}$ for which (3) has a solution. Clearly the system has a solution iff

$$\begin{cases} \frac{1}{3}(22 - x) \leq \frac{2}{5}(x - 11) \\ x \geq 15\,, \end{cases} \tag{4}$$

or equivalently iff

$$\begin{cases} x \geq 16 \\ x \geq 15\,, \end{cases} \tag{5}$$

i.e. iff $16 \leq x$; thus $s = 16$. The sub-replication inequality has the opposite sign, and so we analogously we need to find the largest x for which

$$\begin{cases} h \geq \frac{2}{5}(x - 11) \\ 0 \leq 15 - x \\ h \leq \frac{1}{3}(22 - x)\,; \end{cases} \tag{6}$$

has a solution, which happens iff

$$\begin{cases} x \leq 16 \\ x \leq 15\,, \end{cases} \tag{7}$$

i.e. iff $15 \geq x$; thus $d = 15$.

2nd solution Recall that, if the interval

$$\mathcal{V}_0 := \{\mathbb{E}^{\mathbb{Q}}[\bar{V}_1] : \mathbb{Q} \in \mathcal{M}\}$$

is just a singleton $\{a\}$, then $V_0 = a$ is the unique arbitrage free price of a contract with payoff V_1 and V_1 is replicable, whereas if $\mathcal{V}_0$ is not a singleton then $\mathcal{V}_0 = (a, b)$ for some $a < b$, in which case the set of arbitrage free prices of V_1 is the open interval (a, b) and V_1 is not replicable. So, let us then compute $\mathbb{E}^{\mathbb{Q}}[\bar{X}_1]$ for $\mathbb{Q} \in \mathcal{M}$ as

$$\begin{pmatrix} \frac{6}{11}(1-t) \\ t \\ \frac{5}{11}(1-t) \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 22 \\ 30 \\ 44 \end{pmatrix} = 6(1-t) + 15t + 10(1-t) = 16 - t.$$

and $\mathbb{E}^{\mathbb{Q}}[\bar{Y}_1]$ for $\mathbb{Q} \in \mathcal{M}$ as

$$\begin{pmatrix} \frac{6}{11}(1-t) \\ t \\ \frac{5}{11}(1-t) \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 22 \\ 32 \\ 44 \end{pmatrix} = 6(1-t) + 16t + 10(1-t) = 16.$$

This shows that the set of all arbitrage free prices for X_1 is

$$\{16 - t : t \in (0,1)\} = (15, 16),$$

and for Y_1 is $\{16\}$, and X_1 is not replicable while Y_1 is replicable.

(g) 1^{st} **solution:** Since Y was replicable in the (B,S) market, the set of all possible final wealths in the (B,S,Y) market is the same as in the (B,S) market. As we saw, this only has dimension 2, and thus these markets are not complete.

Said otherwise, the replication equation $fB_1 + gS_1 + hY_1 = W_1$ does not always have a solution for any value of W_1, because it corresponds to a system of 3 equations (one for each ω_i) in 3 variables f, g, h, but the equations are not independent (because Y_1 is a linear combination of B_1 and S_1), and so the system does not always have a solution.

2^{nd} **solution:** Since any EMM $\mathbb{Q}$ for the market (B,S) satisfies $\mathbb{E}[\overline{Y}_1] = \overline{Y}_0$, any such $\mathbb{Q}$ is also an EMM for the market (B,S,Y), and thus the two markets have the same set of EMMs. Since the set of EMMs for (B,S) is not a singleton, the same holds for (B,S,Y), which is thus not complete.

(h,i) **1st solution:** The enlarged market (B,S,X) is arbitrage-free, since $31/2$ is an arbitrage free price for X_1, because it satisfies $31/2 \in (d(X), u(X)) = (15, 16)$. One can also check this directly, looking for g, h such that the discounted final wealth $\overline{V}_1^{0,g,h} = g(\overline{S}_1 - \overline{S}_0) + h(\overline{X}_1 - \overline{X}_0)$ is positive, and finding out that this implies that $\overline{V}_1^{0,g,h} = 0$ by applying the FM algorithm. Indeed, if

$$\begin{cases} g(\frac{-5}{2}) + h(11 - \frac{31}{2}) \geq 0 \\ g(0) + h(15 - \frac{31}{2}) \geq 0 \\ g(3) + h(22 - \frac{31}{2}) \geq 0 \end{cases}$$

then the 2nd equation is equivalent to $h \leq 0$, and isolating the g term the equivalent system

$$\begin{cases} g \leq -\frac{9}{5}h \\ 0 \geq h \\ g \geq -\frac{13}{6}h \end{cases}$$

from which we get $-\frac{9}{5}h \geq -\frac{13}{6}h$ and so $h \geq 0$, which combined with $0 \geq h$ gives $h = 0$, and so $g \in [-\frac{9}{5}h, \frac{13}{6}h]$ also must equal 0, and so $\overline{V}_1^{0,g,h} = 0$.

Moreover, the market (B,S,X) is also complete, since to replicate W_1 one has to solve the replication equation

$$x + h(\overline{S}_1 - \overline{S}_0) + k(\overline{X}_1 - \overline{X}_0) = \overline{W}_1$$

which corresponds to the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} -\frac{5}{2} \\ 0 \\ 3 \end{pmatrix} + k \begin{pmatrix} 11 - 31/2 \\ 15 - 31/2 \\ 22 - 31/2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{8}$$

where $w_i := W_1(\omega_i)$, and this is a system of 3 independent equations in 3 unknowns, so it has a (unique) solution for any value of $w = (w_1, w_2, w_3)$.

2nd solution: If $\mathbb{Q}$ is an EMM for (B, S), then it is an EMM for (B, S, X) iff $\overline{X}_0 = \mathbb{E}^{\mathbb{Q}}[\overline{X}_1]$. Thus we take $t \in (0, 1)$ and ask that

$$q_t := \begin{pmatrix} \frac{6}{11}(1-t) \\ t \\ \frac{5}{11}(1-t) \end{pmatrix} \text{ satisfies } \frac{31}{2} = \frac{6}{11}(1-t) \cdot 11 + t \cdot 15 + \frac{5}{11}(1-t) \cdot 22; \qquad \text{(EMMX)}$$

which admits the unique solution $t = 1/2$. Thus the model (B, S, X) is arbitrage free and complete, since there is a unique EMM.