This document contains 2 questions.

1. [default,O3g]

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. For $t \in \mathbb{R}$, define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	3	5
$X_1(\omega)$	1	3	t

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 0$, a stock whose initial price is $S_0 = 4$, and whose final price is S_1. We also consider a derivative X with payoff X_1. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) free of arbitrage?

A. No **B. Yes**

(b) Find all the values of t (if any) for which X_1 replicable (in the market (B, S)).

A. 8 B. 5 **C. None of the above**

Assume from now on that X has initial cost $X_0 = 4$.

(c) Find all the values of t (if any) for which the market (B, S, X) is free of arbitrage.

A. 5 B. $\frac{11}{2}$ **C.** $(5, \frac{11}{2})$ D. $[5, \frac{11}{2}]$ E. None of the above

Assume from now on that $t = 5$.

(d) Can EMM be used to determine whether the market (B, S, X) is complete?

A. No B. Yes

(e) Find explicitly an arbitrage strategy in the market (B, S, X).

Solution:

(a) 1^{st} **solution:** Yes, since $2/4 = d < 1 + r = 1 < u = 5/4$.

2^{nd} **solution:** It is enough to show that there exists EMMs, as we now do.

Recall that $\mathbb{Q}$ is an EMM (equivalent martingale measure) if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 4 = 2q_1 + 3q_2 + 5q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system of equalities has solution $q_2 = 2 - 3q_3, q_1 = -1 + 2q_3$, and imposing $q_i > 0$ we obtain that the set of (q's corresponding to the set of) EMM is

$$\mathcal{M} = \left\{ q_t := \begin{pmatrix} -1+2s \\ 2-3s \\ s \end{pmatrix} : s \in \left(\frac{1}{2}, \frac{2}{3}\right) \right\}. \tag{1}$$

Since $\mathcal{M}$ is not empty, the model is arbitrage-free.

(b) 1^{st} **solution:** Since X_1 is replicable iff it has a unique arbitrage-free price, let us determine the set $\mathcal{P}$ of arbitrage-free prices of X_1 in the (B, S) market. By the RNPF

$$\mathcal{P} = \{\mathbb{E}^{\mathbb{Q}}[X_1/(1+r)] : \mathbb{Q} \in \mathcal{M}\},$$

and using eq. (1) and substituting the numerical values gives

$$\mathcal{P} = \mathcal{P}(t) := \{-1 + 2s + 3(2-3s) + ts : s \in \left(\frac{1}{2}, \frac{2}{3}\right)\}. \tag{2}$$

Evaluating the above expression at $s = \frac{1}{2}$ and at $s = \frac{2}{3}$ gives

$$\frac{3+t}{2}, \quad \frac{1+2t}{3}.$$

Since the function

$$s \mapsto -1 + 2s + 3(2-3s) + ts$$

is affine, it is either constant or strictly monotone. Thus, $\mathcal{P}(t)$ is a singleton ($= X_1$ is replicable) iff

$$\frac{3+t}{2} = \frac{1+2t}{3},$$

i.e. iff $t = 7$, in which case $\mathcal{P}(7) = \{5\}$. Otherwise, $\mathcal{P}(t)$ is the open interval with endpoints $\frac{3+t}{2}, \frac{1+2t}{3}$.

2^{nd} **solution:** We could look for the values of t for which the replication equation $V_1^{x,h} = X_1$ has a solution x, h, i.e. for the values of t, x, h for which $V_1^{x,h} = X_1$. Since

$$V_1^{x,h} = x + h(S_1 - S_0(1+r)),$$

this leads to the following linear system of three equations in three unknowns

$$\begin{cases} x & +h(2-4) & = 1 \\ x & +h(3-4) & = 3 \\ x & +h(5-4) & = t \end{cases}, \quad \text{i.e.} \begin{cases} x & -2h & = 1 \\ x & -h & = 3 \\ x & +h & = t \end{cases}.$$

This system has a unique solution, easily found as follows. Using only the first 2 equations one gets a system in 2 equations in 2 variables, which is quickly solved to give $x = 5, h = 2$; the third equation then gives $t = 7$. Thus, X_1 is replicable iff $t = 7$.

(c) 1^{st} **solution:** By definition, the portfolio (h, g) is an arbitrage is the corresponding final wealth

$$V_1^{0,h,g} = h(S_1 - S_0(1+r)) + g(X_1 - X_0(1+r)),$$

satisfies $V_1^{0,h,g}(\omega) \geq 0$ for all ω, and not all inequalities are satisfies with equality. So, let us solve the system of inequalities $V_1^{0,h,g} \geq 0$, i.e.

$$\begin{cases} h(2-4) + g(1-4) \geq 0 \\ h(3-4) + g(3-4) \geq 0 \\ h(5-4) + \ g(t-4) \geq 0 \end{cases}$$

by applying the FM algorithm. Isolating the variable h we find i.e. the system

$$\begin{cases} h \quad \leq \quad -\frac{3}{2}g \\ h \quad \leq \quad -g \\ h \geq (4-t)g \end{cases},$$

which leads to

$$\begin{cases} -\frac{3}{2}g \geq (4-t)g \\ \quad -g \geq (4-t)g \end{cases},$$

i.e. to

$$\begin{cases} ((t-5) - \frac{1}{2})g \geq 0 \\ \qquad (t-5)g \geq 0 \end{cases}.$$

There are now 3 cases.

If $t - 5 \geq \frac{1}{2}$ (i.e. $t \geq \frac{11}{2}$) then the solution to the system is $g \geq 0$, in which case the inequality $(t-5)g \geq 0$ is satisfied strictly if $g > 0$. Thus, any $g > 0$ is an arbitrage.

If $\frac{1}{2} > t - 5 > 0$ (i.e. $t \in (5, \frac{11}{2})$) then the only solution to the system is $g = 0$, in which case both inequalities are satisfied with equality, so there is no arbitrage.

If $0 \geq t - 5$ (i.e. $5 \geq t$) then the solution to the system is $g \leq 0$, in which case the inequality $(t - 5 - \frac{1}{2})g \geq 0$ is satisfied strictly if $g < 0$. Thus, any $g < 0$ is an arbitrage.

So, there is no arbitrage iff $t \in (5, \frac{11}{2})$.

2^{nd} **solution:** By definition, (B, S, X) is free of arbitrage iff $X_0 \in \mathcal{P}(t)$. Since $X_0 = 4$, using the expression for $\mathcal{P}(t)$ found in the previous item gives the following. If $t = 7$ then $X_0 = 4 \notin \mathcal{P}(7) = \{5\}$, and if $t > 7$ then $4 < 5 < \frac{3+t}{2} < \frac{1+2t}{3}$, so in both cases (B, S, X) is not free of arbitrage. If $t < 7$ then $\frac{1+2t}{3} < \frac{3+t}{2}$, and then

$$4 \in \mathcal{P}(t) = \left(\frac{1+2t}{3}, \frac{3+t}{2}\right)$$

holds iff $t < \frac{11}{2} = 5.5$ and $t > 5$. Thus (B, S, X) is free of arbitrage iff $t \in (5, \frac{11}{2})$.

3^{rd} **solution:** (B, S, X) is free of arbitrage iff there exists some EMMs in such market.

Recall that $\mathbb{Q}$ is an EMM for (B, S, X) if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, $\bar{X}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{X}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 4 = 2q_1 + 3q_2 + 5q_3 \\ 4 = q_1 + 3q_2 + tq_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system of equalities has solution only if $t \neq 7$, in which case the unique solution is

$$q_1 = \frac{5-t}{t-7}, \quad q_2 = \frac{2t-11}{t-7}, \quad q_3 = \frac{1}{7-t}.$$

Imposing additional the constraint $q_3 > 0$ gives $t < 7$. Given $t < 7$, q_1 holds iff $t > 5$, and $q_2 > 0$ holds iff $2t - 11 < 0$. So, we have found that the model is arbitrage-free if and only if $t < 7, t > 5, t < \frac{11}{2}$, i.e. iff $t \in (5, \frac{11}{2})$.

(d) Since X is not replicable for $t = 5 \neq 7$, the market (B, S, X) is complete, as the system of replication equations is composed of 3 equations in 3 unknowns, and the equations are independent (the first two because S_1 is not constant; the third because X_1 is not replicable in (B, S)). Notice that we cannot use the 2nd fundamental theorem of asset pricing to answer this question, since it assumes that the set of EMM is not empty.

(e) Since $S_0 = X_0, S_1(\omega_i) = X_1(\omega_i)$ for $i = 2, 3$, and $S_1(\omega_1) = 2 > 1 = X_1(\omega_1)$, trivially buying one share of S and selling one unit of X is an arbitrage.

2. [default,O16b]

Consider a one-period arbitrage-free model (B, S) of market composed of a bank account B with interest rate r, and one underlying S. To find the price of a money-back call option, complete the following steps.

(a) Write down a formula for the function f s.t. $f(S_1, m)$ is the payoff of the derivative which, at maturity $T = 1$, provides its buyer with both a call option with strike K on the underlying S, and the amount $m \in \mathbb{R}$ if the call with strike K is in the money (i.e. if $S_T \geq K$).

(b) From now on let (B, S) be given by the binomial model with

$$r = 0, \quad S_0 = 12, \quad S_1(H) = 20, \quad S_1(T) = 4, \quad K = 12.$$

Compute the initial value $X_0(m)$ of the derivative with payoff $X_1(m) := f(S_1, m)$.

(c) In the above binomial model (B, S), compute the price M_0 of the *money-back call* option, which at maturity $T = 1$ gives its buyer a call option on S with strike $K = 12$, plus it repays the initial cost M_0 if the call with strike K finished in the money.

What is the value of M_0?

A. 10 **B. 8** C. 4 D. None of the above

Solution:

(a)

$$f(s,m) := (s-K)^+ + m1_{[k,\infty)}(s) = \begin{cases} s-k+m & \text{if } s \geq k \\ 0 & \text{if } s < k. \end{cases}, \quad \text{defined for } s,m \in \mathbb{R}.$$

(b) **1st solution:** To find the replicating portfolio (x,h) for such derivative, we must solve the equation

$$x(1+r) + h(S_T - S_0) = (S_T - K)^+ + m1_{\{S_T \geq k\}}.$$

Since the payoff to be replicated is

$$f(S_1(H),m) = (20-12)^+ + m1_{[12,\infty)}(20) = 8+m, \quad f(S_1(T),m) = (4-12)^+ + m1_{[12,\infty)}(4) = 0,$$

the replication equation becomes the following system of equations

$$\begin{cases} x+8h &= 8+m \\ x-8h &= 0, \end{cases} \tag{3}$$

which has solution $h = \frac{1}{2} + \frac{1}{16}m$, $x = 4 + \frac{1}{2}m$, and so $X_0(m) = x = 4 + \frac{1}{2}m$.

2nd solution: We first determine the EMM

$$u := S_1(H)/S_0 = 5/3, \quad d := S_1(T)/S_0 = 1/3, \quad \mathbb{Q}(H) = \frac{(1+r)-d}{u-d} = \frac{1}{2}.$$

and then we apply the RNPF to find that the option has the unique arbitrage-free price

$$X_0(m) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[X_1(m)\right] = \frac{1}{2}(8+m) + (1-\frac{1}{2})(0) = 4 + \frac{1}{2}m.$$

(c) M_0 is determined by asking that the initial cost of the option with payoff $f(S_T, M_0)$ is M_0, i.e. M_0 is the value of m which solves the equation $X_0(m) = m$. Since we found that $X_0(m) = 4 + \frac{1}{2}m$, this gives $M_0 = 8$.