This document contains 4 questions.

1. [default,Q9]

   Consider on $\Omega = \{H, T\}^2$ the rv Y, X, Z defined as follows:

ω	$Y(\omega)$	$Z(\omega)$	$X(\omega)$
HH	9	1	1
HT	6	2	2
TH	6	2	3
TT	3	4	4

   Is X $\sigma(Y)$-measurable? Is Z $\sigma(Y)$-measurable?

   A. No, No **B. No, Yes** C. Yes, No D. Yes, Yes

Solution:

Is X $\sigma(Y)$-measurable? Let us answer in three different ways

- No: X is not constant on $\{Y = 6\} = \{HT, TH\}$ ($X(HT) \neq X(TH)$), i.e. knowing $Y = 6$ does not determine the value of X
- No: $\nexists h | X = h \circ Y : Y(HT) = 6 = Y(TH)$ so $h(Y)(HT) = h(6) = h(Y)(TH)$ but $X(HT) \neq X(TH)$.
- No: find $F \in \mathcal{B}(\mathbb{R})$ s.t. $X^{-1}(F) \notin \sigma(Y)$, $\quad X^{-1}(\{2\}) = \{HT\} \notin \sigma(Y)$, since $\{HT\}$ is not a union of sets of the form $\{Y = y_n\}$.

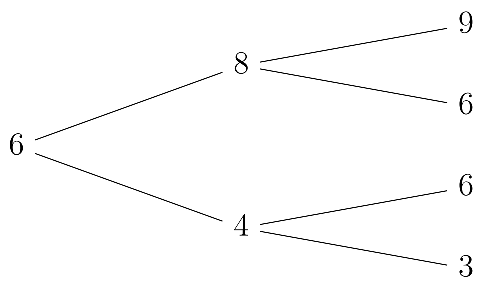
Z is $\sigma(Y)$-measurable? Let us answer in three different ways:

- Yes: Z is constant on $\{HT, TH\}$ since $Z(HT) = 2 = Z(TH)$, i.e. knowing $Y = 6$ determines that the value of Z is 2
- Yes: $Z = h(Y)$ where h is the function given by $h(9) = 1, h(6) = 2, h(3) = 4$.
- Yes: $Z^{-1}(F) \in \mathcal{F}$ for any $F \in \{\{1\}, \{2\}, \{4\}\}$, and thus for any $F \subseteq \{1, 2, 4\}$.

Notice that we considered $\tilde{\Omega} = \{3, 6, 9\}$, but we could have taken $\tilde{\Omega} = \mathbb{R}$ (or $\tilde{\Omega} = \mathbb{N}$), in which case we have to define h (arbitrarily) on $\tilde{\Omega} \setminus \{3, 6, 9\}$.

2. [default,Q12]

   Let S defined on the binomial space $\Omega = \{H, T\}^2 = \{HH, HT, TH, TT\}$ be given by the binary tree



Write down each element of the 4 σ-algebras $\sigma(S_1, S_2)$,$(\sigma(S_i))_{i=0,1,2}$. How many elements k does $\sigma(S_2)$ have?

A. $k \leq 3$ B. $k = 5$ C. $k = 6$ **D.** $k = 8$ E. None of the above

Solution: As always, $\sigma(S_i)$ is the family of sets of the form $\{S_i \in A\}$ for measurable $A \subseteq \mathbb{R}$. Since S takes values in $\tilde{\Omega} := \{8, 4, 3, 6, 9\}$, it is enough to consider $A \subseteq \tilde{\Omega}$; this observation is convenient, since $\tilde{\Omega}$ is finite, so every every element of $\sigma(S_i)$ is a union of elements of the form $\{S_i = a\}$ for some $a \in A \subseteq \tilde{\Omega}$. So, we get that

$$\sigma(S_0) = \{\emptyset, \Omega\}, \qquad S_0^{-1}(A) = \begin{cases} \emptyset & 6 \notin A \\ \Omega & 6 \in A \end{cases}$$

$$S_1^{-1}(A) = \{S_1 \in A\} = \begin{cases} \Omega, & 4, 8 \in A \\ \emptyset, & 4, 8 \notin A \\ \{S_1 = 4\} = \{TH, TT\}, & 4 \in A, 8 \notin A \\ \{S_1 = 8\} = \{HH, HT\}, & 4 \notin A, 8 \in A \end{cases}$$

and so $\sigma(S_1) = \{\Omega, \emptyset, \{S_1 = 4\}, \{S_1 = 8\}\}$, and in particular $\sigma(S_1) = \sigma(X_1)$, as it was intuitive.

Analogously it is intuitive that $\sigma(S_1, S_2) = \sigma(X_1, X_2) = \mathcal{P}(\Omega)$; to prove this, just notice that any path $\omega = (\omega_1, \omega_2)$ can be fully described by the values $(S_1(\omega), S_2(\omega))$, so choosing $A = \{(S_1(\omega), S_2(\omega))\}$ gives $(S_1, S_2)^{-1}(A) = \{\omega\}$, i.e. given any $\omega \in \Omega$ the set $\{\omega\}$ is in $\sigma(S_1, S_2)$, and so $\sigma(S_1, S_2) = \mathcal{P}(\Omega)$.

Finally $\sigma(S_2)$ equals

$$\{\Omega, \emptyset, \{S_2 = 3\}, \{S_2 = 6\}, \{S_2 = 9\}, \{S_2 = 6 \text{ or } 9\}, \{S_2 = 3 \text{ or } 9\}, \{S_2 = 3 \text{ or } 6\}\},$$

since (to clarify with an example) $\{S_2 = 6 \text{ or } 9\} = \{S_2 \in A\}$ for any $A \subseteq \tilde{\Omega}$ s.t. $3 \notin A, 6, 9 \in A$ etc.

We can explicitly write the elements of $\sigma(S_2)$ as follows:

$$\{S_2 = 9\} = \{HH\}, \quad \{S_2 = 6\} = \{HT, TH\}, \quad \{S_2 = 3\} = \{TT\}$$

which leads to

$$\{S_2 = 6 \text{ or } 9\} = \{S_2 = 6\} \cup \{S_2 = 9\} = \{HH, HT, TH\},$$

and analogously $\{S_2 = 3 \text{ or } 9\} = \{HH, TT\}$, $\{S_2 = 3 \text{ or } 6\} = \{TT, HT, TH\}$.

3. [default,Q17]

Endow the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the natural filtration $\mathcal{F} := \mathcal{F}^X$ generated by a process $(X_t)_{t\in\mathbb{T}}$, with finite time index $\mathbb{T} := \{0, 1, \ldots, T\}$. Let Y be a non-constant random variable independent of X, and define the filtration $\mathcal{G}$ by taking $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(Y) := \sigma(\mathcal{F}_t \cup \sigma(Y))$. Consider the processes:

1. $$A_t := \begin{cases} X_0^2 + \sum_{s=1}^{t+1}(X_s - X_{s-1})^2 & \text{if } t \in \mathbb{T}, t < T \\ 0 & \text{if } t = T \end{cases}$$
2. $B_t := X_0^2 + \sum_{s=0}^{t-1}(X_{s+1} - X_s)^2$, $t \in \mathbb{T}$
3. $C_t := X_0^2 + \sum_{s=1}^{t-1}(X_s - X_{s-1})^2$, $t \in \mathbb{T}$
4. $D_t := Y + A_t$, $t \in \mathbb{T}$
5. $E_t := YB_t$, $t \in \mathbb{T}$
6. $F_t := \exp(Y)C_t$, $t \in \mathbb{T}$

For each of the following questions, select all correct answers.
Hint: A rv which is independent by itself must be a.s. constant.

(a) Which of the processes A, B, C are $\mathcal{F}$-adapted?
A. A B. A, B **C.** B, C D. C

(b) Which of the processes A, B, C are $\mathcal{F}$-predictable?
A. A B. A, B C. B, C **D.** C

(c) Which of the processes D, E, F are $\mathcal{G}$-adapted?
A. D B. D, E **C.** E, F D. F

Solution: Y cannot be $\mathcal{F}_T$-measurable, otherwise, since it is independent of $\mathcal{F}_T$, it would be independent of itself, yet Y is not constant. A process W is adapted to a filtration $\mathcal{H}$ $\iff$ W_t is $\mathcal{H}_t$-measurable for every t, and predictable iff W_t if $\mathcal{H}_{t-1}$-measurable for all t. Since X_t is $\mathcal{F}_t$-measurable but not $\mathcal{F}_{t-1}$-measurable, and Y is $\mathcal{G}_t$-measurable for all t but not $\mathcal{F}_t$-measurable for any t, the answers follow.

Here a proof that a rv W which is independent by itself must be constant. By definition of independence

$$\mathbb{E}[f(W)g(W)] = \mathbb{E}[f(W)]\mathbb{E}[g(W)], \quad \forall f, g,$$

so in particular $\mathbb{E}[W]^2 = \mathbb{E}[W^2]$, and so simple algebra gives $\mathbb{E}[(W - \mathbb{E}[W])^2] = 0$, so $(W - \mathbb{E}[W])^2 = 0$ a.s., and so $W = \mathbb{E}[W]$ a.s..

4. [default,O2]

Consider the following one period trinomial model: $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\omega_i) = 1/3$ for $i = 1, 2, 3$, a bank account B with interest rate $r = 0$, and one stock S with $S_0 = 6$ and

$$S_1(\omega) = \begin{cases} 2, & \text{if } \omega = \omega_1, \\ 6, & \text{if } \omega = \omega_2, \\ 12, & \text{if } \omega = \omega_3. \end{cases}$$

We denote with $C(K)$ the European call option (on the stock) with strike price $K \geq 0$; this has payoff $C_1(K) := (S_1 - K)^+$ at time 1. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) arbitrage free?

A. No **B. Yes**

(b) Is the call option $C(K_1)$ with strike $K_1 = 4$ replicable?

A. No B. Yes

(c) What is the set $\mathcal{P}$ of arbitrage free prices (at time 0, in the given market (B, S)) of a call option with strike $K_1 = 4$?

A. $\{2\}$ B. $\{\frac{16}{5}\}$ **C.** $(2, \frac{16}{5})$ D. $[2, \frac{16}{5}]$ E. None of the above

(d) Consider the enlarged market $(B, S, C(K_1))$ made of: bank account, stock, call option with strike $K_1 = 4$ sold at time 0 at an arbitrage-free price $C_0(4) \in \mathcal{P}$. Is this market complete?

A. No **B. Yes** C. not enough info (the answer depends on $C_0(4)$)

(e) Enlarge the market $(B, S, C(4))$ considered in the previous item with call options with strike $K_2 = 5$, sold at time 0 at price $C_0(K_2)$. We do not assume that $C_0(5)$ is necessarily an arbitrage free price; instead we assume that $C_0(K_2)$ satisfies the inequalities

$$C_0(K_2) \leq C_0(K_1) \leq C_0(K_2) + K_2 - K_1 \tag{A}$$

$$(S_0 - K_2)^+ \leq C_0(K_2) \leq S_0. \tag{B}$$

It can be shown that in any market model where at least one of these inequalities fails there is an arbitrage. Does the converse hold, i.e. do our assumptions *imply* that the enlarged market $(B, S, C(K_1), C(K_2))$ is arbitrage free? If yes, prove it; if not, explicitly find values of $C_0(K_1), C_0(K_2)$ which satisfy (A), (B) and for which the market admits an arbitrage.

A. No B. Yes

Solution:

(a) 1^{st} **solution:** This trinomial model is free of arbitrage since the condition $d < 1+r < u$ is satisfied: indeed $d = 2/6 = 1/3$, $1 = r = 1$, $u = 12/6 = 2$.

2^{nd} **solution:** Recall that $\mathbb{Q}$ is an Equivalent Martingale Measure (EMM) if $S_0 = \mathbb{E}^{\mathbb{Q}}[S_1/(1+r)]$, $\mathbb{Q}$ is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 6 = 2q_1 + 6q_2 + 12q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system has 2 equalities and 3 unknowns, so it has one free parameter. So we choose $q_2 = t$, and compute q_1, q_3 as $q_1 = 3(1-t)/5, q_3 = 2(1-t)/5$, and imposing $q_i > 0$ we obtain that the set $\mathcal{M}$ of (q's corresponding to the set of) EMMs is

$$\mathcal{M} := \left\{ \mathbb{Q}_t : (\mathbb{Q}_t(\omega_i))_i := q_t := \begin{pmatrix} 3(1-t)/5 \\ t \\ 2(1-t)/5 \end{pmatrix} \text{ for } t \in (0, 1) \right\}. \tag{1}$$

As $\mathcal{M}$ is not empty, this confirms that the model is arbitrage-free, thanks to the FTAP.

(b) In the market (B, S) the call with strike K_1 is not replicable.

1^{st} **solution:** This follows from the fact that, since $r = 0$, the replication equation is $x + h(S_1 - S_0) = C(K_1)$, which in vector notation becomes

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} 2-6 \\ 6-6 \\ 12-6 \end{pmatrix} = \begin{pmatrix} (2-4)^+ \\ (6-4)^+ \\ (12-4)^+ \end{pmatrix},$$

i.e.

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix}, \tag{2}$$

which has no solution. Indeed its first equation gives $x = 4h$, its first equation gives $x = 2$, combining these gives $h = 1/2$, and these values do not solve the third equation since $2 + 6/2 = 5$ does not equal 8.

2^{nd} **solution:** The replicability criterion says that $C_1(4)$ is replicable if and only if it has a unique AFP (Arbitrage Free Price). Since

$$C_1(4) = \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix},$$

the set of AFP is given by

$$\mathcal{P} := \left\{ \frac{\mathbb{E}^{\mathbb{Q}}[C_1(4)]}{1+r} : \mathbb{Q} \in \mathcal{M} \right\} = \left\{ \frac{3}{5}(1-t) \cdot 0 + t \cdot 2 + \frac{2}{5}(1-t) \cdot 8 = \frac{16}{5} - \frac{6}{5}t, \quad t \in (0,1) \right\} \tag{3}$$

which is obviously not a singleton (because $\frac{16}{5} - \frac{6}{5}t$ is not constant in $t \in (0,1)$), so $C_1(4)$ is not replicable.

(c) **1st solution:** In this simple setting, the less computationally intensive way to find its AFP is probably to compute the smallest super-replication price s (and largest sub-replication price i). To do that, we replace the equality $\overline{V}_1^{x,h} = C_1(K)$ with the inequality $\overline{V}_1^{x,h} \geq C_1(K)$; then s is the smallest x for which such system has a solution. To find it, we then replace eq. (2) with the system

$$\begin{cases} x - 4h \geq 0 \\ x \geq 2 \\ x + 6h \geq 8 \end{cases}$$

and to eliminate the variable h to we rewrite this as

$$\begin{cases} \frac{1}{4}x \geq h \\ x - 2 \geq 0 \\ -\frac{1}{6}x + \frac{4}{3} \leq h \end{cases}.$$

This leads to the system in x

$$\begin{cases} \frac{1}{4}x & \geq -\frac{1}{6}x + \frac{4}{3} \\ x - 2 & \geq 0 \end{cases},$$

whose solution is any $x \geq 16/5$; the smallest such x is thus $s = 16/5$. Analogously i is the largest in x s.t. $\overline{V}_1^{x,h} \leq C_1(K)$, i.e. s.t.

$$\begin{cases} \frac{1}{4}x & \leq -\frac{1}{6}x + \frac{4}{3} \\ x - 2 & \leq 0 \end{cases},$$

i.e. $i = 2$. Thus the set of arbitrage-free prices is $(2, \frac{16}{5}) = (2, 3.2)$.

2nd solution: The set of AFP can be computed explicitly: by eq. (3) it equals the range $g((0,1)) := \{g(t) : t \in (0,1)\}$ of the function $(0,1) \ni t \mapsto g(t) := \frac{16}{5} - \frac{6}{5}t$, and since $g(0) = \frac{16}{5} > g(1) = 2$ and g is affine (and thus continuous, and either constant or strictly monotone), its range is $g((0,1)) = (2, \frac{16}{5}) = (2, 3.2)$.

(d) **1st solution:** Whether the market is complete not never depends on the initial value of the traded assets (and so in particular of $C_0(K_1)$): indeed, completeness of a one-period market $(S^1, \dots, S^m)$ means that for every payoff D_1 there exists variables $h^1, \dots, h^m \in \mathbb{R}$ such that the replication equation

$$kB_1 + \sum_{j=1}^{m} h^j S_1^j = D_1$$

has a solution, and such equation does not depend on the initial values $(S_0^1, \dots, S_0^m)$. This is less obvious (but of course equally true) when the replication equation is written in terms of the initial capital $x = k + \sum_{j=1}^{m} h^j S_0^j$, since in this case the replication equation is

$$x + \sum_{j=1}^{m} h^j (S_1^j - S_0^j) = D_1$$

which does depend on S_0^j.

This specific market $(B, S, C(K_1))$ is complete. Indeed, the replication equation

$$k + hS_1 + gC_1(K_1) = D_1$$

for a derivative with payoff D_1 corresponds to the system of equations

$$kv_1 + hv_2 + gv_3 = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix} + k \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \tag{4}$$

where $d_i := D_1(\omega_i)$. which always has a solution. Indeed, the system has 3 unknowns k, h, g, and is made of independent equations, because the vectors v_1, v_2, v_3 which represent the payoff of bank account, stock and call option, are linearly independent: indeed v_1, v_2 are linearly independent (one is not a multiple of the other), and v_3 is not a linear combination of v_1, v_2 (otherwise $C_1(4)$ would have been replicable). Another method is to prove directly that the vectors v_1, v_2, v_3 are linearly independent by calculating the rank of the matrix M that has v_1, v_2, v_3 as its columns, and showing that it is 3. Since M is a square matrix, whose determinant can be easily calculated as

$$(-2)(12-2) + 8(6-2) = 12,$$

showing that it is non-zero, it has rank 3.

2nd solution: For an alternative solution, observe that, for any choice of AFP $C_0(4)$, the market $(B, S, C(4))$ has only one EMM (thus it is complete), which is the unique EMM $\mathbb{Q}$ for the (B, S) market s.t.

$$C_0(4) = \mathbb{E}^{\mathbb{Q}}[C_1(4)/(1+r)]. \tag{5}$$

Such $\mathbb{Q}$ is unique because the equation $C_0(4) = \frac{16}{5} - \frac{6}{5}t$ has a unique solution (for any $C_0(4) \in \mathcal{P}$), indeed the solution is $t = \frac{5}{6}(\frac{16}{5} - C_0(4))$.

(e) Since the market $(B, S, C(K_1))$ is complete, any derivative has a *unique* AFP in $(B, S, C(K_1))$. The idea is that, since by assumption $C_0(K_2)$ satisfies the inequalities

$$C_0(K_2) \leq C_0(K_1) \leq C_0(K_2) + K_2 - K_1 \tag{6}$$
$$(S_0 - K_2)^+ \leq C_0(K_2) \leq S_0, \tag{7}$$

which do not fix uniquely the exact value of $C_0(K_2)$ but only require it to be in some interval I, all but at most one value of $C_0(K_2) \in I$ will result in an arbitrage.

Of course, we actually have to explicitly build the counter-example to check that I is not degenerate (i.e. a singleton), which would invalidate our argument above; let us do that.

1st solution: We can choose any AFP for $C_0(K_1)$ in the (B, S) market; since $(B, S, C(K_1))$ is complete, it has only one EMM, which is the unique EMM $\mathbb{Q}$ for the (B, S) market s.t. (5) holds. We then use such $\mathbb{Q}$ to find the unique AFP p for $C_1(5)$ in the $(B, S, C(K_1))$ market, and show that there is a value of $C_0(5)$ which satisfies (6) , (7) and yet is different from p.

Instead of choosing $C_0(K_1)$ and finding $\mathbb{Q}$, it is easier to work backwards: choose a $\mathbb{Q} \in \mathcal{M}$, then take $C_0(K_1)$ as given (5).

So, we choose a value of $t \in (0, 1)$, say $t = \frac{1}{2}$, which fixes $\mathbb{Q} \in \mathcal{M}$, and we take

$$C_0(K_1) := \frac{1}{2} \cdot 2 + \frac{2}{5}(1 - \frac{1}{2}) \cdot 8 = \frac{13}{5}.$$

With this as value of $C_0(K_1)$, the only EMM for the market $(B, S, C(K_1))$ is the one given by (1) with $t = \frac{1}{2}$. Since the payoff $C_1(5)$ is given by

$$\begin{pmatrix} (2-5)^+ \\ (6-5)^+ \\ (12-5)^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix},$$

the only AFP for $C_1(5)$ in the $(B, S, C(K_1))$ market is given by

$$p = \frac{3}{5}(1 - \frac{1}{2}) \cdot 0 + \frac{1}{2} \cdot 1 + \frac{2}{5}(1 - \frac{1}{2}) \cdot 7 = \frac{19}{10}.$$

Plugging in the values of $K_2, K_1, C_0(4)$ shows that (6) , (7) become

$$\frac{13}{5} - 1 \leq C_0(K_2) \leq \frac{13}{5}, \qquad (6-5)^+ \leq C_0(K_2) \leq 6,$$

or equivalently that $\frac{8}{5} \leq C_0(K_2) \leq \frac{13}{5}$. Thus, with our choice of $C_0(K_1) = \frac{13}{5}$, any value of $C_0(5) \in [\frac{8}{5}, \frac{13}{5}] \setminus \{\frac{19}{10}\}$ (for example $C_0(5) = 2$) satisfies eqs. (6) and (7), yet the corresponding market $(B, S, C(K_1), C(K_2))$ has an arbitrage.

2nd solution: Alternatively, we can just to try out a few choices of values and see if any of them works. A natural educated guess is to take $C_0(4) = C_0(5) = p$, where p is some values in the set $\mathcal{P} = (2, 16/5)$ of arbitrage-free prices for $C_0(4)$. Indeed, obviously the strategy which buys one option with strike 4 and short-sells one with strike 5 is an arbitrage, since it has cost $C_0(4) - C_0(5) = 0$ and final payoff

$$C_1(4) - C_1(5) = (S_1 - 4)^+ - (S_1 - 5)^+ \geq 0,$$

which is not identically zero (it is strictly positive when $S_1 = 6$ and when $S_1 = 12$). So, it remains to prove that these values satisfy eqs. (6) and (7). Trivially eq. (6) is satisfied when $C_0(4) = C_0(5) = p$, since $K_2 \geq K_1$. To show that eq. (7) holds with $C_0(4) = C_0(5) = p$ we need to show that

$$C_0(K_1) \leq S_0, \quad S_0 - K_2 \leq C_0(K_1), \quad 0 \leq C_0(K_1).$$

The inequality $C_0(K_1) \leq S_0$ holds, since $S_1 \geq 0$ implies $C_1(4) = (S_1 - 4)^+ \leq S_1$, and so the domination principle (which holds in the market $(B, S, C(4))$, since it is assumed to be arbitrage-free) implies $C_0(4) \leq S_0$. Analogously $S_0 - K_2 \leq C_0(K_1)$ follows from the domination principle and the chain of inequalities

$$S_1 - K_2(1+r) \leq S_1 - K_1(1+r) = S_1 - K_1 \leq (S_1 - K_1)^+ = C_1(K_1),$$

which show that the value of the portfolio made of one call with strike 4 dominates the one made of one stock and which borrows K_2 from the bank. Analogously $0 \leq C_0(K_1)$ follows from the domination principle and $0 \leq (S_1 - K_1)^+ = C_1(K_1)$.