This document contains 4 questions.

1. [default,Q9]

   Consider on $\Omega = \{H, T\}^2$ the rv Y, X, Z defined as follows:

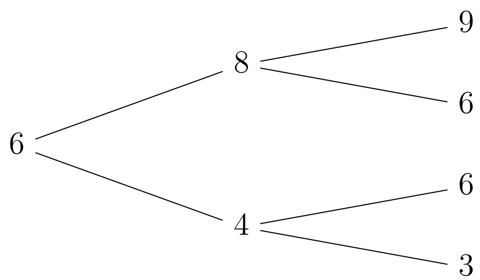
ω	$Y(\omega)$	$Z(\omega)$	$X(\omega)$
HH	9	1	1
HT	6	2	2
TH	6	2	3
TT	3	4	4

   Is X $\sigma(Y)$-measurable? Is Z $\sigma(Y)$-measurable?

   A. No, No B. No, Yes C. Yes, No D. Yes, Yes

2. [default,Q12]

   Let S defined on the binomial space $\Omega = \{H, T\}^2 = \{HH, HT, TH, TT\}$ be given by the binary tree



   Write down each element of the 4 σ-algebras $\sigma(S_1, S_2)$,$(\sigma(S_i))_{i=0,1,2}$. How many elements k does $\sigma(S_2)$ have?

   A. $k \leq 3$ B. $k = 5$ C. $k = 6$ D. $k = 8$ E. None of the above

3. [default,Q17]

   Endow the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the natural filtration $\mathcal{F} := \mathcal{F}^X$ generated by a process $(X_t)_{t \in \mathbb{T}}$, with finite time index $\mathbb{T} := \{0, 1, \dots, T\}$. Let Y be a non-constant random variable independent of X, and define the filtration $\mathcal{G}$ by taking $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(Y) := \sigma(\mathcal{F}_t \cup \sigma(Y))$. Consider the processes:

   1. $$A_t := \begin{cases} X_0^2 + \sum_{s=1}^{t+1}(X_s - X_{s-1})^2 & \text{if } t \in \mathbb{T}, t < T \\ 0 & \text{if } t = T \end{cases}$$
   2. $B_t := X_0^2 + \sum_{s=0}^{t-1}(X_{s+1} - X_s)^2$, $t \in \mathbb{T}$
   3. $C_t := X_0^2 + \sum_{s=1}^{t-1}(X_s - X_{s-1})^2$, $t \in \mathbb{T}$
   4. $D_t := Y + A_t$, $t \in \mathbb{T}$
   5. $E_t := Y B_t$, $t \in \mathbb{T}$
   6. $F_t := \exp(Y) C_t$, $t \in \mathbb{T}$

   For each of the following questions, select all correct answers.
   Hint: A rv which is independent by itself must be a.s. constant.

   (a) Which of the processes A, B, C are $\mathcal{F}$-adapted?

   A. A B. A, B C. B, C D. C

(b) Which of the processes A, B, C are $\mathcal{F}$-predictable?

A. A B. A, B C. B, C D. C

(c) Which of the processes D, E, F are $\mathcal{G}$-adapted?

A. D B. D, E C. E, F D. F

4. [default,O2]

Consider the following one period trinomial model: $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\omega_i) = 1/3$ for $i = 1, 2, 3$, a bank account B with interest rate $r = 0$, and one stock S with $S_0 = 6$ and

$$S_1(\omega) = \begin{cases} 2, & \text{if } \omega = \omega_1, \\ 6, & \text{if } \omega = \omega_2, \\ 12, & \text{if } \omega = \omega_3. \end{cases}$$

We denote with $C(K)$ the European call option (on the stock) with strike price $K \geq 0$; this has payoff $C_1(K) := (S_1 - K)^+$ at time 1. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) arbitrage free?

A. No B. Yes

(b) Is the call option $C(K_1)$ with strike $K_1 = 4$ replicable?

A. No B. Yes

(c) What is the set $\mathcal{P}$ of arbitrage free prices (at time 0, in the given market (B, S)) of a call option with strike $K_1 = 4$?

A. $\{2\}$ B. $\{\frac{16}{5}\}$ C. $(2, \frac{16}{5})$ D. $[2, \frac{16}{5}]$ E. None of the above

(d) Consider the enlarged market $(B, S, C(K_1))$ made of: bank account, stock, call option with strike $K_1 = 4$ sold at time 0 at an arbitrage-free price $C_0(4) \in \mathcal{P}$. Is this market complete?

A. No B. Yes C. not enough info (the answer depends on $C_0(4)$)

(e) Enlarge the market $(B, S, C(4))$ considered in the previous item with call options with strike $K_2 = 5$, sold at time 0 at price $C_0(K_2)$. We do not assume that $C_0(5)$ is necessarily an arbitrage free price; instead we assume that $C_0(K_2)$ satisfies the inequalities

$$C_0(K_2) \leq C_0(K_1) \leq C_0(K_2) + K_2 - K_1 \tag{A}$$

$$(S_0 - K_2)^+ \leq C_0(K_2) \leq S_0. \tag{B}$$

It can be shown that in any market model where at least one of these inequalities fails there is an arbitrage. Does the converse hold, i.e. do our assumptions *imply* that the enlarged market $(B, S, C(K_1), C(K_2))$ is arbitrage free? If yes, prove it; if not, explicitly find values of $C_0(K_1), C_0(K_2)$ which satisfy (A), (B) and for which the market admits an arbitrage.

A. No B. Yes