This document contains 2 questions.

Question 1

(Total: 0 marks)

[default,M11]

Consider a *quanto call* option, which has payoff $X_T = (E_0 S_T - k)^+$ at maturity T, where S is the value of a foreign asset in the foreign currency, the strike price k is set in the domestic currency, and E_0 is the exchange rate between currencies at time 0 (so that $E_0 S_0$ is the value at time 0 of the foreign asset in the domestic currency).

You are a British investor and have to compute the price (in £) X_0 at time 0 of a quanto call option with strike £8, whose underlying, a stock traded in Germany, is modelled as having the following values in € at maturity $N = 2$

ω	HH	HT	TH	TT
$S_2(\omega)$	1	13/4	4	9

Suppose the domestic (i.e. for £) interest rate is the constant $r = \frac{1}{4} = 25\%$, the foreign (i.e. for €) interest rate is the constant $q = 1 = 100\%$, and the exchange rate E between £ and € (defined as the cost of one € in £) is modelled as

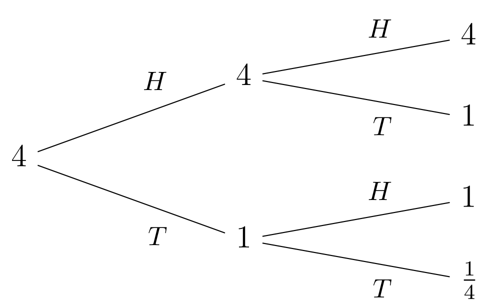

Figure 1: Tree of E.

Consider the market composed of the domestic and foreign bank accounts, and the foreign asset.

(a) Are there values of S_0, S_1 for which there is no arbitrage ?

A. No **B. Yes**

(b) To what interval of values does the unique arbitrage-free price X_0 of X at time 0 belong to?

A. $(-\infty, 0]$ B. $(0, 3)$ C. $[3, 4)$ D. $[4, 6)$ **E. None of the above**

Solution:

1. What is the market to be considered? Let us see how we can invest. We could either deposit £ in a (British) bank account with interest r, or use £ to buy € which, since we are a British investor, are to be considered as just another financial asset, for which we always have to consider its value Y_n at time n in £. Since a € deposited in a (German) bank account accumulate interest at rate q, the value Y_n of a € in £ is $Y_n := E_n(1+q)^n$. Analogously, we can invest in the foreign asset,

whose values in £ is given by $W := ES$. Thus, we have reduced the problem to pricing a derivative with payoff X_2 at time 2 in the 2-period binomial model with two underlying Y, W and interest rate $r = 1/4$.

The market (B, Y) has no arbitrage, since the up and down factors u, d of Y satisfy $1/2 = d < 1 + r = \frac{5}{4} < u = 2$, and it is complete, being a binomial model. Equivalently, there exists one and only one EMM $\mathbb{Q}$, and it is given by

$$\mathbb{Q}(\omega_i|\mathcal{F}_n) = \tilde{p}_n = \frac{(1+r_n) - d_n}{u_n - d_n} = \frac{\frac{5}{4} - \frac{2}{4}}{\frac{4}{2} - \frac{1}{2}} = \frac{1}{2}.$$

Thus, the payoff $W_2 = E_2 S_2$ is replicable in the model (B, Y), and so there are unique values W_0, W_1 such that (B, Y, W) is arbitrage-free, which can be calculated uniquely with the RNPF

$$\mathbb{E}^{\mathbb{Q}}[\frac{W_n}{1 + r_{n-1}}|\mathcal{F}_{n-1}] = W_{n-1}.$$

Once calculated W_1, W_0, we can the compute $S_n = W_n/E_n$. *So, there exists such values of S_0, S_1 (and you do not need to compute them explicitly to know they exist).*

Let us carry out such computations explicitly (although you were not required to do so, and you won't need these values in the next item). So, we can calculate the values of W_1 as

$$W_1(H) = \frac{4}{5} \cdot \frac{1}{2}(4 + \frac{13}{4}) = \frac{29}{10}, \quad W_1(T) = \frac{4}{5} \cdot \frac{1}{2}(4 + \frac{9}{4}) = \frac{25}{10} = \frac{5}{2}$$

and combining those we get

$$W_0 = \frac{4}{5} \cdot \frac{1}{2}(\frac{29}{10} + \frac{25}{10}) = \frac{54}{25},$$

so that the market is

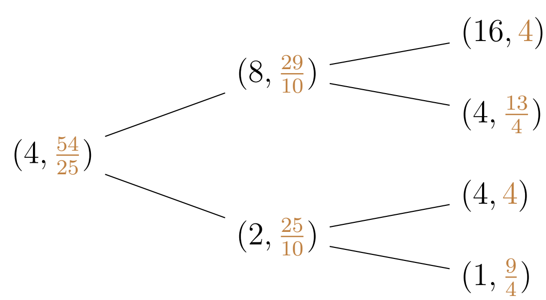

Figure 2: Tree of (Y, W).

Finally we compute

$$S_1(H) = \frac{W_1(H)}{E_1(H)} = \frac{\frac{29}{10}}{4} = \frac{29}{40}, \quad S_1(T) = \frac{W_1(T)}{E_1(T)} = \frac{\frac{25}{10}}{1} = \frac{5}{2}, \quad S_0 = \frac{W_0}{E_0} = \frac{\frac{54}{25}}{4} = \frac{27}{50}.$$

2. The value $X_2 = (E_0 S_2 - k)^+ = (4S_2 - 8)^+$ in £ of the quanto at maturity 2 is

ω	HH	HT	TH	TT
$X_2(\omega)$	0	5	8	28

We can use the RNPF, using the previously calculated value of $\tilde{p}_n = \frac{1}{2}$, to compute

$$X_1(\omega) = \frac{1}{1+r}(\tilde{p}_1(\omega)X_2(\omega H) + (1-\tilde{p}_1)(\omega)X_2(\omega T)) = \frac{4}{5}\cdot\frac{1}{2}\left(X_2(\omega H) + X_2(\omega T)\right),$$

and so

$$X_1(H) = \frac{4}{5}\cdot\frac{1}{2}(0+5) = 2, \quad X_1(T) = \frac{4}{5}\cdot\frac{1}{2}(8+28) = \frac{72}{5}.$$

and analogously

$$X_0 = \frac{1}{1+r}(\tilde{p}_0 X_1(H) + (1-\tilde{p}_0)X_1(T)) = \frac{4}{5}\cdot\frac{1}{2}(X_1(H) + X_1(T)) = \frac{4}{5}\cdot\frac{1}{2}\left(2 + \frac{72}{5}\right) = \frac{164}{25}.$$

Question 2

(Total: 0 marks)

[default,M20]

Model a risky asset S with a 2-period binomial model with constant up factor 2 and down factor 1/2 and initial value $S_0 = 4$, and a bank account with constant interest rate $r = 1/4$.

1. What is the price at time 0 of the American put option on S with strike price $K = 5$?
   A. $\frac{2}{5}$ **B.** $\frac{34}{25}$ C. $\frac{24}{25}$ D. None of the above
2. Let τ^* be the smallest optimal exercise time. Which of the following statements about τ^* are correct?
   A. $\tau^*(HH) = 2$ **B.** $\tau^*(HT) = 2$ **C.** $\tau^*(TH) = 1$ D. $\tau^*(TT) = 2$

Solution:

(a) At every time step, the buyer of the American put can choose to receive for payment the intrinsic value of the put $I_n := K - S_n$, or to wait so (s)he can exercise later. Thus, the value V_2 at time 2 of the American put which hasn't been exercised at previous times is I_2 if one exercises immediately, and is 0 if one chooses not to exercise at all, and so $V_2 := I_2^+ := I_2 \vee 0$. The value P_1 at time 1 of receiving a payment of V_2 at time 2 is given by the RNPF $P_1 = \mathbb{E}[\frac{I_2}{1+r}|\mathcal{F}_1]$. Thus, the value V_1 at time 1 of the American put which hasn't been exercised at previous times is $V_1 = I_1 \vee P_1 := \max(I_1, P_1)$, because at time 1 of owner of the option can choose whether to get immediately paid the amount I_1, or whether to wait, in which case at time 2 he will own a derivative which is worth V_2 at time 2, and thus is worth P_1 at time 1.

We then proceed analogously to go from time 1 to time 0. In other words, for any American derivative, the value V_n at time n of the American option with intrinsic value I_n which has not been previously exercised is given by the following algorithm: at maturity we can either exercise the option and get $I_N := K - S_N$, or we can choose not to exercise at all and thus get 0, so $V_N := I_N^+ = I_N \vee 0$ and then one can compute V by backward induction via the formula

$$V_n := \max\left(I_n, \mathbb{E}[\frac{V_{n+1}}{1+r}|\mathcal{F}_n]\right), n = 0, 1, \ldots, N-1.$$

Let us now carry out the numerical calculations in detail. The intrinsic value $I := 5 - S$ of the put option with strike 5 is displayed in the following tree

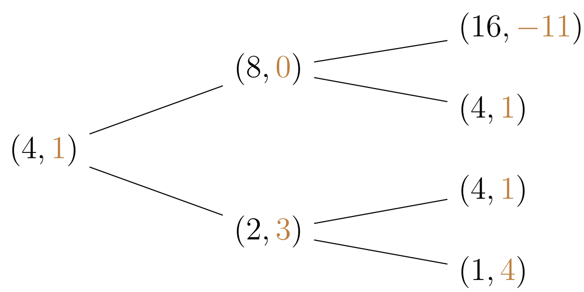

Figure 3: Tree of $(S, 5 - S)$.

Thus, the above algorithm leads to the following calculations. First, take $V_2 = I_2^+ := (5 - S_2)^+$. Then compute the value P_1 at time 1 of receiving the payment V_2 at time 2 by using the RNPF. Since $r = \frac{1}{4}, u = 2, d = 1/2$ we get $\tilde{p} = 1/2$ and so

$$\frac{\tilde{p}}{1+r} = \frac{2}{5} = \frac{1-\tilde{p}}{1+r},$$

we get

$$P_1(H) = \frac{2}{5}(0+1) = \frac{2}{5}, \quad P_1(T) = \frac{2}{5}(1+4) = 2.$$

Then we set $V_1 = \max(P_1, I_1)$, and so we get

$$V_1(H) = \frac{2}{5}, \quad V_1(T) = 3.$$

Analogously, we compute the value P_0 at time 0 of receiving the payment V_1 at time 1 using the RNPF

$$P_0 = \frac{2}{5}(\frac{2}{5} + 3) = \frac{34}{25}, \quad \text{and then} \quad V_0 = \max(P_0, I_0) = \max(\frac{34}{25}, 1) = \frac{34}{25}.$$

(b) At time 0 it is best not to exercise the option, since the value of exercising immediately is I_0, whereas the value at time 0 of waiting (until at least time 1) is $P_0 > I_0$. At time 1, if the first coin toss was a tail then $I_1(T) = 3 > 2 = P_1(T)$, so it is best to exercise immediately. If instead $X_1 = H$ then $I_1(H) = 0 < \frac{2}{5} = P_1(H)$, so it is best to wait; in this case at time 2 if $X_2 = H$, if we exercise the put we get $K - S_2(HH) = -11 < 0$, and so we choose not to exercise; whereas if $X_2 = T$, if we exercise the put we get $K - S_2(HT) = 1 > 0$, and so we choose to exercise. In summary

ω	HH	HT	TH	TT
$\tau^*(\omega)$	∞	2	1	1

Notice that if $K - S_2(HT)$ had been equal to 0 (e.g. if we had considered $K = 4$) then we would have been indifferent between exercising at time 2, or waiting further (i.e. not exercising at all, since 2 was the last time at which we could exercise), in which case there would have been two possible optimal exercise times.

This reasoning shows that, in any discrete time binomial model, a stopping time τ is an optimal exercise time for an American option with intrinsic value I iff

$$\{\tau = n\} \subseteq \{V_n = I_n\}, \quad \text{for all } n = 0, \dots, N,$$

and so the smallest optimal exercise time is

$$\tau^* := \inf\{n \in \{0, \dots, N\} : V_n = I_n\},$$

where as usual we use the convention $\inf \emptyset := \infty$ and V_n denoted the value at time n of the American option which hasn't yet been exercised.