This document contains 2 questions.

1. [default,Q18]

Let $X, Y$ be IID rvs with Bernoulli distribution of parameter $p \in (0,1)$, i.e.

$$P(X=1)=p, \quad P(X=0)=1-p, \quad \text{and define } Z := 1_{\{X+Y=0\}}.$$

Compute $E[X|Z]$ and $E[Y|Z]$ for arbitrary $p \in (0,1)$, then answer:

(a) If $p = 1/3$, which values does $E[X|Z]$ take?

A. $0, 1/3$ B. $0, 3/8$ **C. $0, 3/5$** D. None of the above

(b) Are $E[X|Z]$ and $E[Y|Z]$ independent?

*Hint: A rv which is independent by itself must be constant.*

A. Yes, always (for any $p \in (0,1)$)

B. It depends on the value of $p \in (0,1)$

**C. Never (for no $p \in (0,1)$)**

**Solution:**

(a) The random variable $Z$ takes only the two values 1 and 0, and

$$\{Z=0\} = A_0 := \{X=0\} \cap \{Y=0\}, \quad \{Z=1\} = A_1 := \{X=1\} \cup \{Y=1\}$$

respectively. Thus

$$\mathbb{E}[X|Z](\omega) = \begin{cases} a_0 & \text{if } \omega \in A_0 \\ a_1 & \text{if } \omega \in A_1 \end{cases}$$

where $a_i = \mathbb{E}[X 1_{A_i}]/\mathbb{P}(A_i)$. Since $X = 0$ on $A_0$ we get $a_0 = 0$. Since $\{X = 1\} \subseteq A_1$ we get that

$$1_{A_1} X = 1_{A_1}(1 \cdot 1_{\{X=1\}} + 0 \cdot 1_{\{X=0\}}) = 1_{A_1} 1_{\{X=1\}} = 1_{\{X=1\}}. \tag{1}$$

Since $X, Y$ are independent, the complement $A_0$ of $A_1$ has probability

$$\mathbb{P}(A_0) = \mathbb{P}(\{X=0\} \cap \{Y=0\}) = \mathbb{P}(\{X=0\})\mathbb{P}(\{Y=0\}) = (1-p)^2,$$

from which, using (1), it follows that

$$a_1(p) = \frac{\mathbb{E}[X 1_{A_1}]}{\mathbb{P}(A_1)} = \frac{\mathbb{E}[1_{\{X=1\}}]}{1 - \mathbb{P}(A_0)} = \frac{p}{1-(1-p)^2}.$$

If $p = 1/3$ then

$$a_1 = \frac{p}{1-(1-p)^2} = \frac{\frac{1}{3}}{1-\frac{4}{9}} = \frac{3}{5},$$

so

$$\mathbb{E}[X|Z](\omega) = \begin{cases} 0 & \text{if } \omega \in A_0 \\ 3/5 & \text{if } \omega \in A_1 \end{cases}$$

(b) By symmetry $\mathbb{E}[X|Z] = \mathbb{E}[Y|Z]$. Let us prove by contradiction that $\mathbb{E}[X|Z], \mathbb{E}[Y|Z]$ cannot be independent. If they were, we would have that $W := \mathbb{E}[X|Z]$ is independent of itself, yet $W$ is not constant (since $a_0 \neq a_1(p)$ for all $p \in (0,1)$), contradiction.

2. [default,P14]

Let $c \neq 0$ be a constant, $(X_i)_{i\in\mathbb{N}}$ be IID rvs with the same law as the rv $X$, and for $n \in \mathbb{N}\setminus\{0\}$ set

$$S_0 := 0, \quad S_n := \sum_{i=1}^{n} X_i, \quad Q_0 := 0, \quad Q_n := \sum_{i=1}^{n} X_i^2, \quad \mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_n := \sigma(X_1, \ldots, X_n).$$

Calculate the values of the constants $a, b, d, y, z$ such that the following processes $A, B, C, D, Z$ are martingales:

(a) $A_n := S_n - an$ for $n \in \mathbb{N}$, assuming $\mathbb{E}|X| < \infty$.
A. $a = \frac{1}{2}\mathbb{E}[X]$ **B.** $a = \mathbb{E}[X]$ C. $a = 2\mathbb{E}[X]$ D. $a = 0$ E. None of the above

(b) $B_n := Q_n - bn$ for $n \in \mathbb{N}$, assuming $\mathbb{E}X^2 < \infty$.
A. $b = \frac{1}{2}\mathbb{E}[X^2]$ **B.** $b = \mathbb{E}[X^2]$ C. $b = 2\mathbb{E}[X^2]$ D. $b = 0$ E. None of the above

(c) $C_n := \exp(cS_n - nd)$ for $n \in \mathbb{N}$, assuming $|X| \leq c < \infty$.
A. $d = \mathbb{E}[\exp(cX)]$ **B.** $d = \log(\mathbb{E}[\exp(cX)])$ C. $d = \log(\mathbb{E}[cX])$ D. $d = \mathbb{E}[\log(cX)]$ E. $d = 0$

(d) $D_n := Y_n - yn$ for $n \in \mathbb{N}$, where $Y_n := |S_{n\wedge\tau}|$ for $\tau := \inf\{n \geq 1 : S_n = 0\}$, assuming $P(X = \pm 1) = 1/2$.
*Hint: Show that $\{\tau \leq n\}$ is $\mathcal{F}_n$-measurable, write $1 = 1_{\{\tau\leq n\}} + 1_{\{\tau>n\}}$ and give explicit expressions for $Y_{n+1}$ as a function of $(Y_n, X_{n+1})$ on $\{\tau \leq n\}$, and on $\{\tau > n\}$.*
A. $y = 4$ B. $y = 1$ C. $y = \frac{1}{4}$ **D.** $y = 0$ E. None of the above

(e) $Z$ defined by: $Z_0 := 1, Z_{n+1} := zZ_n/2^{X_{n+1}}$ for $n \in \mathbb{N}$, assuming $\mathbb{P}(X = k) = 1/2^k$ for $k \in \mathbb{N}\setminus\{0\}$.
**A.** $z = 3$ B. $z = 1$ C. $z = \frac{1}{3}$ D. None of the above

**Solution:**

(a) Since

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_n + X_{n+1}|\mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1}]$$

we get

$$\mathbb{E}[A_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_{n+1}|\mathcal{F}_n] - a(n+1) = S_n - na - a + \mathbb{E}[X_{n+1}],$$

so $A$ is a martingale iff $a = \mathbb{E}[X]$.

(b) Since

$$\mathbb{E}[Q_{n+1}|\mathcal{F}_n] = \mathbb{E}[Q_n + X_{n+1}^2|\mathcal{F}_n] = Q_n + \mathbb{E}[X_{n+1}^2],$$

$B$ is a martingale iff $b = \mathbb{E}[X^2]$.

(c) Since

$$\mathbb{E}[\exp(cS_{n+1})|\mathcal{F}_n] = \mathbb{E}[\exp(cS_n)\exp(cX_{n+1})|\mathcal{F}_n] = \exp(cS_n)\mathbb{E}[\exp(cX_{n+1})],$$

$C$ is a martingale iff $d = \log(\mathbb{E}[\exp(cX)])$.

(d) Notice that $\{\tau \le n\}$ is $\mathcal{F}_n$-measurable since it equals $\cup_{i=1}^n \{S_i = 0\}$. On the set $\{\tau \le n\}$ we have $Y_n = 0 = Y_{n+1}$, whereas on $\{\tau > n\}$ we have $S_n \neq 0, Y_n \neq 0$, and since $X_{n+1}$ can only takes values $\pm 1$ and $S$ has values in $\mathbb{N}$, we get that if $S_n > 0$ then $S_{n+1} \ge 0$, and so

$$Y_n = S_n, \quad Y_{n+1} = S_{n+1} = S_n + X_{n+1} = Y_n + X_{n+1};$$

whereas if $S_n < 0$ then $S_{n+1} \le 0$ and so

$$Y_n = -S_n, \quad Y_{n+1} = -S_{n+1} = -S_n - X_{n+1} = Y_n - X_{n+1}.$$

Summing up, if $\text{sign}(s)$ denotes the sign of $s$, we get that on $\{\tau > n\}$

$$Y_{n+1} = Y_n + \text{sign}(S_n) X_{n+1} = \begin{cases} Y_n + X_{n+1} & \text{if } S_n > 0, \\ Y_n - X_{n+1} & \text{if } S_n < 0. \end{cases}$$

Thus we see that

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[Y_{n+1} 1_{\{\tau \le n\}}|\mathcal{F}_n] + \mathbb{E}[Y_{n+1} 1_{\{\tau > n\}}|\mathcal{F}_n]$$

equals

$$\mathbb{E}[1_{\{\tau>n\}}(Y_n + \text{sign}(S_n)X_{n+1})|\mathcal{F}_n] = 1_{\{\tau>n\}} Y_n + 1_{\{\tau>n\}} \text{sign}(S_n)\mathbb{E}[X_{n+1}|\mathcal{F}_n].$$

Since $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1}] = 0$ and $1_{\{\tau>n\}} Y_n = Y_n$ (because $Y_n = 0$ on $\{\tau \le n\}$) we get $\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n$, i.e. $Y$ is a martingale, so $D$ is a martingale if $y = 0$.

(e) Since

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = z Z_n \mathbb{E}[1/2^{X_{n+1}}|\mathcal{F}_n] = z Z_n \mathbb{E}[1/2^{X_{n+1}}],$$

$Z$ is a martingale iff $z = 1/\mathbb{E}[1/2^{X_{n+1}}]$. To compute this explicitly, let us recall how to sum the geometric series: since

$$(1-c)\sum_{n=1}^{k} c^n = \sum_{n=1}^{k} c^n - \sum_{n=2}^{k+1} c^n = c - c^{k+1},$$

for $c \in (0,1)$ we have that $\sum_{n=1}^k c^n \uparrow \frac{c}{1-c}$ as $k \to \infty$. It follows that

$$\mathbb{E}[\frac{1}{2^X}] = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{1}{2^k} = \frac{1}{4} + \frac{1}{4^2} + \ldots = \frac{1/4}{1-1/4} = \frac{1}{3},$$

and so $Z$ is a martingale iff $z = 3$.