This document contains 2 questions.

1. [default,Q18]

   Let X, Y be IID rvs with Bernoulli distribution of parameter $p \in (0,1)$, i.e.

   $$P(X=1) = p, \quad P(X=0) = 1-p, \quad \text{and define } Z := 1_{\{X+Y=0\}}.$$

   Compute $E[X|Z]$ and $E[Y|Z]$ for arbitrary $p \in (0,1)$, then answer:

   (a) If $p = 1/3$, which values does $E[X|Z]$ take?

   A. $0, 1/3$ B. $0, 3/8$ C. $0, 3/5$ D. None of the above

   (b) Are $E[X|Z]$ and $E[Y|Z]$ independent?

   Hint: A rv which is independent by itself must be constant.

   A. Yes, always (for any $p \in (0,1)$)

   B. It depends on the value of $p \in (0,1)$

   C. Never (for no $p \in (0,1)$)

2. [default,P14]

   Let $c \neq 0$ be a constant, $(X_i)_{i\in\mathbb{N}}$ be IID rvs with the same law as the rv X, and for $n \in \mathbb{N} \setminus \{0\}$ set

   $$S_0 := 0, \quad S_n := \sum_{i=1}^{n} X_i, \quad Q_0 := 0, \quad Q_n := \sum_{i=1}^{n} X_i^2, \quad \mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_n := \sigma(X_1, \ldots, X_n).$$

   Calculate the values of the constants a, b, d, y, z such that the following processes A, B, C, D, Z are martingales:

   (a) $A_n := S_n - an$ for $n \in \mathbb{N}$, assuming $\mathbb{E}|X| < \infty$.

   A. $a = \frac{1}{2}\mathbb{E}[X]$ B. $a = \mathbb{E}[X]$ C. $a = 2\mathbb{E}[X]$ D. $a = 0$ E. None of the above

   (b) $B_n := Q_n - bn$ for $n \in \mathbb{N}$, assuming $\mathbb{E}X^2 < \infty$.

   A. $b = \frac{1}{2}\mathbb{E}[X^2]$ B. $b = \mathbb{E}[X^2]$ C. $b = 2\mathbb{E}[X^2]$ D. $b = 0$ E. None of the above

   (c) $C_n := \exp(cS_n - nd)$ for $n \in \mathbb{N}$, assuming $|X| \leq c < \infty$.

   A. $d = \mathbb{E}[\exp(cX)]$ B. $d = \log(\mathbb{E}[\exp(cX)])$ C. $d = \log(\mathbb{E}[cX])$ D. $d = \mathbb{E}[\log(cX)]$ E. $d = 0$

   (d) $D_n := Y_n - yn$ for $n \in \mathbb{N}$, where $Y_n := |S_{n\wedge\tau}|$ for $\tau := \inf\{n \geq 1 : S_n = 0\}$, assuming $P(X = \pm 1) = 1/2$.

   Hint: Show that $\{\tau \leq n\}$ is $\mathcal{F}_n$-measurable, write $1 = 1_{\{\tau\leq n\}} + 1_{\{\tau>n\}}$ and give explicit expressions for Y_{n+1} as a function of (Y_n, X_{n+1}) on $\{\tau \leq n\}$, and on $\{\tau > n\}$.

   A. $y = 4$ B. $y = 1$ C. $y = \frac{1}{4}$ D. $y = 0$ E. None of the above

   (e) Z defined by: $Z_0 := 1, Z_{n+1} := zZ_n/2^{X_{n+1}}$ for $n \in \mathbb{N}$, assuming $\mathbb{P}(X = k) = 1/2^k$ for $k \in \mathbb{N} \setminus \{0\}$.

   A. $z = 3$ B. $z = 1$ C. $z = \frac{1}{3}$ D. None of the above