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1. [default,P12]

   (a) Prove that a random variable which is independent by itself must be a.s. constant.

   (b) Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sigma-algebra $\mathcal{G} \subseteq \mathcal{A}$. Given $A \in \mathcal{F}$, define $B := \{\mathbb{E}[1_A|\mathcal{G}] = 0\}$ (meaning $B := \{\omega : \mathbb{E}[1_A|\mathcal{G}](\omega) = 0\}$). Show that $\mathbb{P}(A \cap B) = 0$.

   (c) Consider rv X, Y, Z s.t. (X, Z) has the same law as (Y, Z); in particular X and Y have the same law μ. Show that for any (Borel, bounded) function f

      i. $\mathbb{E}[f(X)|Z] = \mathbb{E}[f(Y)|Z]$ P a.s.

      ii. Define h_1, h_2 via
      $$h_1(X) := \mathbb{E}[f(Z)|X], \qquad h_2(Y) := \mathbb{E}[f(Z)|Y].$$
      Show that $h_1 = h_2$ μ a.s. (here h_1, h_2 are looked at as random variables defined on the space $\Omega := \mathbb{R}$ endowed with the probability μ).

   (d) Show that if $T_1, \ldots, T_n$ are IID and integrable (meaning $\mathbb{E}[|T_i|] < \infty$) and $T := T_1 + \ldots + T_n$ then $(T_1, T), \ldots, (T_n, T)$ have the same law. Conclude that $\mathbb{E}[T_1|T] = T/n$ by using the results of item (c). Then, compute $\mathbb{E}[T|T_1]$.
   Hint: consider $Z := T_2 + \ldots + T_n$.

Solution:

(a) If W was independent of itself, i.e.
$$\mathbb{E}[f(W)g(W)] = \mathbb{E}[f(W)]\mathbb{E}[g(W)], \quad \forall f, g,$$
then in particular $\mathbb{E}[W]^2 = \mathbb{E}[W^2]$, and so $\mathbb{E}[(W - \mathbb{E}[W])^2] = \mathbb{E}[W^2] - \mathbb{E}[W]^2$ equals 0. Thus $(W - \mathbb{E}[W])^2$ is a positive random variable with 0 expectation, so it is 0 a.s., i.e. $W = \mathbb{E}[W]$ a.s., so W is a.s. constant.

(b) Since $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]]$ for any rv X, and since $B \in \mathcal{G}$, we have
$$\mathbb{E}[1_A 1_B] = \mathbb{E}[\mathbb{E}[1_A 1_B|\mathcal{G}]] = \mathbb{E}[\mathbb{E}[1_A|\mathcal{G}]1_B] \tag{1}$$
Since by definition $\mathbb{E}[1_A|\mathcal{G}] = 0$ on B, we have that $\mathbb{E}[1_A|\mathcal{G}]1_B = 0$, and so by (1) and since $1_{A\cap B} = 1_A 1_B$ we have $\mathbb{P}(A \cap B) = \mathbb{E}[1_A 1_B] = 0$

(c) i. By definition of conditional expectation for any fn g
$$\mathbb{E}[g(Z)\mathbb{E}[f(X)|Z]] = \mathbb{E}[g(Z)f(X)], \qquad \mathbb{E}[g(Z)\mathbb{E}[f(Y)|Z]] = \mathbb{E}[g(Z)f(Y)].$$
Moreover, since (X, Z) has the same law as (Y, Z),
$$\mathbb{E}[g(Z)f(X)] = \mathbb{E}[g(Z)f(Y)],$$

and to we get that $W := \mathbb{E}[f(X)|Z] - \mathbb{E}[f(Y)|Z]$ satisfies $\mathbb{E}[g(Z)W] = 0$ for all g. Since W is $\sigma(Z)$-measurable, there exists a fn h s.t. $W = h(Z)$; but then choosing $g = h$ we get that $\mathbb{E}[h(Z)^2] = \mathbb{E}[g(Z)W] = 0$, and so $W = h(Z) = 0$ a.s., i.e. $\mathbb{E}[f(X)|Z] = \mathbb{E}[f(Y)|Z]$ a.s. (to be precise, one should observe that if f is bounded, then W is bounded, and so h can also chosen to be bounded).

ii. Working like in the previous item we get that

$$\mathbb{E}[g(X)\mathbb{E}[f(Z)|X]] = \mathbb{E}[g(Y)\mathbb{E}[f(Z)|Y]];$$

we cannot simply conclude like before, since $g(X) \neq g(Y)$. Instead, just notice that

$$\mathbb{E}[g(X)\mathbb{E}[f(Z)|X]] = \mathbb{E}[g(X)h_1(X)] = \int_{\mathbb{R}} g(x)h_1(x)\mu(dx)$$

and analogously for $\mathbb{E}[g(X)\mathbb{E}[f(Z)|X]]$, and so

$$\int_{\mathbb{R}} gh_1 d\mu = \int_{\mathbb{R}} gh_2 d\mu.$$

We can now conclude like before: taking $g = h_1 - h_2$ shows that $\int (h_1 - h_2)^2 d\mu = 0$ and so $h_1 = h_2$ μ a.s..

(d) By symmetry it is intuitively clear that $(T_1, T), \ldots, (T_n, T)$ have the same law, i.e. $(T_1, T) \sim (T_i, T)$ for all i; let us now prove it formally. For convenience of notation we only consider the case $i = 2$, though the proof is essentially identical for any i. Since $Z := T_2 + \ldots + T_n$ is independent of T_1 (as it is a function of $T_2, \ldots, T_n$), the independence lemma tells us that, given an arbitrary (Borel, bounded) function h,

$$\mathbb{E}h(T_1, T) = \mathbb{E}h(T_1, T_1 + Z) = \mathbb{E}\Big(\mathbb{E}\left(h\left(T_1, T_1 + Z\right)|T_1\right)\Big) = \mathbb{E}\left(g_1\left(T_1\right)\right)$$

where g_1 is the function

$$g_1(t) := \mathbb{E}h\left(t, t + Z\right) = \mathbb{E}h\left(t, t + T_2 + T_3 + \ldots + T_n\right).$$

The same calculation gives

$$\mathbb{E}h\left(T_2, T\right) = \mathbb{E}\left(g_2\left(T_2\right)\right), \text{ where } g_2(t) := \mathbb{E}h\left(t, t + T_1 + T_3 + \ldots + T_n\right).$$

Since the $(T_i)_i$ are IID, $(T_2, T_3, \ldots, T_n)$ has the same law as $(T_1, T_3, \ldots, T_n)$, and so $g := g_1 = g_2$. Since $(T_i)_i$ are identically distributed, $\mathbb{E}\left(g\left(T_1\right)\right) = \mathbb{E}\left(g\left(T_2\right)\right)$, and so $(T_1, T) \sim (T_2, T)$ follows from

$$\mathbb{E}h\left(T_1, T\right) = \mathbb{E}\left(g\left(T_1\right)\right) = \mathbb{E}\left(g\left(T_2\right)\right) = \mathbb{E}h\left(T_2, T\right).$$

From the previous exercise it follows that $\mathbb{E}[T_1|T] = \ldots = \mathbb{E}[T_n|T]$ a.s.. Therefore

$$n\mathbb{E}[T_1|T] = \mathbb{E}[T_1|T] + \ldots + \mathbb{E}[T_n|T] = \mathbb{E}[T_1 + \ldots + T_n|T] = \mathbb{E}[T|T] = T$$

and so $\mathbb{E}[T_1|T] = T/n$. Finally, since Z and T_1 are independent and $Z + T_1 = T$ we get

$$\mathbb{E}[T|T_1] = \mathbb{E}[Z + T_1|T_1] = \mathbb{E}[Z|T_1] + \mathbb{E}[T_1|T_1] = \mathbb{E}[Z] + T_1 = (n-1)\mathbb{E}[T_1] + T_1.$$