This document contains 1 questions.

1. [default,P12]

   (a) Prove that a random variable which is independent by itself must be a.s. constant.

   (b) Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sigma-algebra $\mathcal{G} \subseteq \mathcal{A}$. Given $A \in \mathcal{F}$, define $B := \{\mathbb{E}[1_A|\mathcal{G}] = 0\}$ (meaning $B := \{\omega : \mathbb{E}[1_A|\mathcal{G}](\omega) = 0\}$). Show that $\mathbb{P}(A \cap B) = 0$.

   (c) Consider rv X, Y, Z s.t. (X, Z) has the same law as (Y, Z); in particular X and Y have the same law μ. Show that for any (Borel, bounded) function f

      i. $\mathbb{E}[f(X)|Z] = \mathbb{E}[f(Y)|Z]$ P a.s.

      ii. Define h_1, h_2 via
      $$h_1(X) := \mathbb{E}[f(Z)|X], \qquad h_2(Y) := \mathbb{E}[f(Z)|Y].$$
      Show that $h_1 = h_2$ μ a.s. (here h_1, h_2 are looked at as random variables defined on the space $\Omega := \mathbb{R}$ endowed with the probability μ).

   (d) Show that if $T_1, \ldots, T_n$ are IID and integrable (meaning $\mathbb{E}[|T_i|] < \infty$) and $T := T_1 + \ldots + T_n$ then $(T_1, T), \ldots, (T_n, T)$ have the same law. Conclude that $\mathbb{E}[T_1|T] = T/n$ by using the results of item (c). Then, compute $\mathbb{E}[T|T_1]$.

      Hint: consider $Z := T_2 + \ldots + T_n$.