This document contains 2 questions.

1. [default,M1]

Consider a binomial market model $(B_n, S_n)_{n\leq N}$ where the bank account B has interest rate $r = 0$ and the price of the underlying S starts at $S_0 = 80$, and its value increases by 10 in case of Heads and decreases by 10 in case of Tails, i.e.

$$S_{n+1}(\omega) := \begin{cases} S_n(\omega_1, \dots, \omega_n) + 10, & \text{if } \omega_{n+1} = H \\ S_n(\omega_1, \dots, \omega_n) - 10, & \text{if } \omega_{n+1} = T \end{cases}, \qquad n \in 0, \dots, N-1.$$

Denote with $\mathbb{Q}$ the unique risk-neutral measure and with X_n the coin tosses, given as usual by $X_n(\omega) = \omega_n$.

(a) Draw the binary tree representing S. Can you draw it as a recombinant tree?

A. No **B. Yes**

(b) Are $(X_n)_{n\leq N}$ independent under $\mathbb{Q}$?

A. No **B. Yes**

(c) Is S Markov under $\mathbb{Q}$?

A. No **B. Yes**

(d) Are $(X_n)_{n\leq N}$ identically distributed under $\mathbb{Q}$?

A. No **B. Yes**

(e) Compute $\mathbb{Q}(\{\omega\})$ for every $\omega \in \{H,T\}^N$, then choose the correct statement

A. $\mathbb{Q}(\{\omega\})$ it not constant in N, nor in $\omega \in \{H,T\}^N$

B. $\mathbb{Q}(\{\omega\})$ it not constant in N, but is constant in $\omega \in \{H,T\}^N$

C. $\mathbb{Q}(\{\omega\}) = 1/2^N$ for all $N \geq 1, \omega \in \{H,T\}^N$

D. None of the above

(f) Consider the following methods to compute the price C_0 of a call option on S with strike $K = 80$ and maturity N.

1. Compute $\mathbb{Q}$, then use it to compute $C_0 = \mathbb{E}^{\mathbb{Q}}[C_N]$, where $C_N := (S_N - 80)^+$
2. Compute C_N , then use $C_n = \mathbb{E}^{\mathbb{Q}}[C_{n+1}|\mathcal{F}_n]$ to compute $(C_n)_{n=0}^{N-1}$ by backward induction

Which of the following statement (about computing C_0 numerically using a computer for big N, say $N > 100$) is correct?

A. Both methods allow to compute C_0 even for big N

B. Only the first method allows to compute C_0 even for big N

C. Only the second method allows to compute C_0 even for big N

D. Neither method allows to compute C_0 for big N

(g) Which of the following statement about computing C_0 by hand when $N = 5$ and using one of two methods above is correct?

A. Both methods allow to compute C_0 reasonably fast

B. Only the first method allows to compute C_0 reasonably fast

C. Only the second method allows to compute C_0 reasonably fast

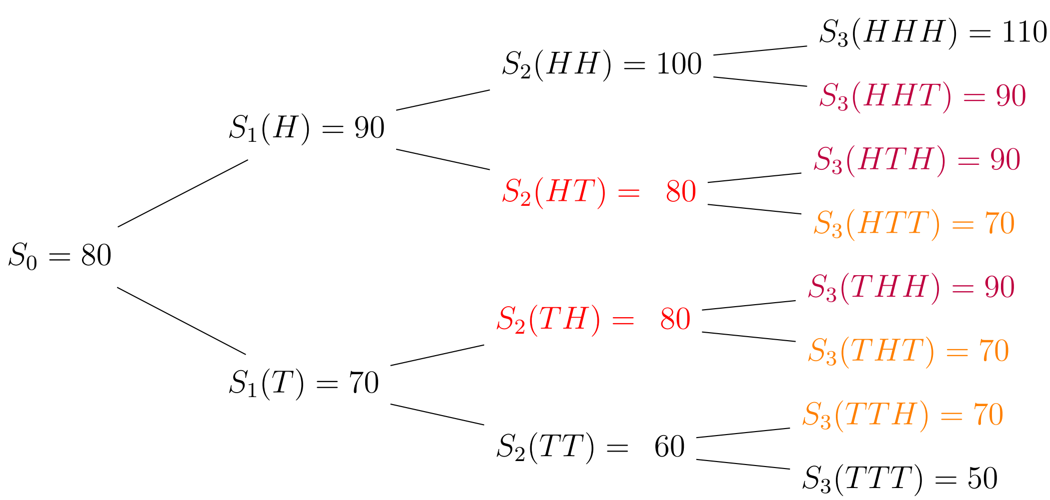
D. Neither method allows to compute C_0 reasonably fast

(h) Compute C_0 by hand when $N = 5$, then choose the correct statement

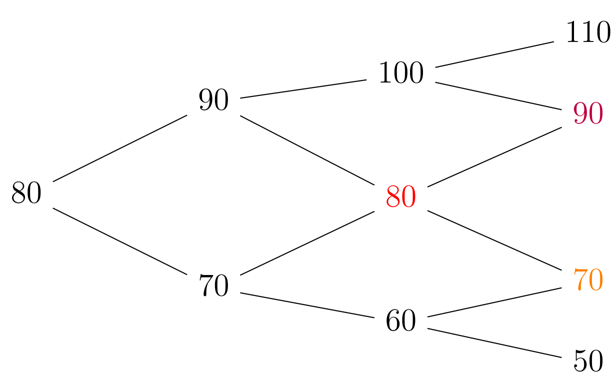
A. $C_0 \in (6, 8)$ **B.** $C_0 \in [8, 10]$ C. $C_0 \in (10, 12)$ D. None of the above

Solution:

(a) $S_n(\omega(n)) = 80 + 10H_n(\omega(n)) - 10T_n(\omega(n))$, where $H_n(\omega) := H_n(\omega(n)) := \sum_{k=1}^n 1_{\{H\}}(X_k)$ equals the numbers of Heads in the first n coin tosses, and $T_n(\omega) = \sum_{k=1}^n 1_{\{T\}}(X_k) = n - H_n(\omega(n))$ the number of tails (while this is clear, you can prove it by induction if you care). Thus S is permutation-invariant, since it depends only on the numbers of Heads in the first n coin tosses, not on the order with which they came out. So, S can be represented by a recombinant tree. This can also be guesses drawing the binary tree of $(S_n)_{n \leq N}$ up to time $N = 3$, which is



which can be represented by a recombining tree as follows



(b) Writing $S_n := S_n(\omega_1, \dots, \omega_n)$ we see that the up and down factors U_n, D_n are

$$\frac{S_{n+1}}{S_n}(\omega_1, \dots, \omega_n, \omega_{n+1}) = \begin{cases} U_n & \text{if } \omega_{n+1} = H \\ D_n & \text{if } \omega_{n+1} = T \end{cases} = \begin{cases} \dfrac{S_n + 10}{S_n} & \text{if } \omega_{n+1} = H \\ \dfrac{S_n - 10}{S_n} & \text{if } \omega_{n+1} = T \end{cases}. \tag{1}$$

As usual, U, D are adapted, i.e. U_n, D_n can depend on $(\omega_1, \dots, \omega_n)$ but not on ω_i for $i \geq n+1$ (since they are functions of $S_n(\omega_1, \dots, \omega_n)$). The corresponding risk neutral probabilities are

$$\tilde{P}_n = \frac{1 + r - D_n}{U_n - D_n} = \frac{1 - \frac{S_n - 10}{S_n}}{\frac{S_n+10}{S_n} - \frac{S_n-10}{S_n}} = \frac{1}{2}, \quad \tilde{Q}_n = 1 - \tilde{P}_n = \frac{1}{2}.$$

Thus, in this particular case $\tilde{P}_n$ is actually deterministic (i.e. constant in ω), which shows that the coin tosses are independent under $\mathbb{Q}$, since

$$\tilde{P}_n(\omega_1, \dots, \omega_n) := \mathbb{Q}(X_{n+1} = H | (X_1, \dots, X_n) = (\omega_1, \dots, \omega_n)). \tag{2}$$

(c) As usual, to show that S is $\mathbb{Q}$-Markov, we try to write S_{n+1} as a function of S_n (which is $\mathcal{F}_n$-measurable, since it only depends on the first n coin tosses, i.e. it is a function of $(X_1, \dots, X_n)$), and a rv B_n which is independent (under $\mathbb{Q}$) from $\mathcal{F}_n$, and then apply the independence lemma to get that $\mathbb{E}^{\mathbb{Q}}[f(S_{n+1})|\mathcal{F}_n]$ equals $g(S_n)$ for some g. While in general we do this by taking $B_n := S_{n+1}/S_n$, here instead we take $B_n := S_{n+1} - S_n$, and thus we write $S_{n+1} = S_n + B_n$. Indeed, since X_{n+1} is independent of $\mathcal{F}_n$, the identity $S_{n+1} - S_n = h(X_{n+1})$ (where h is the function $h(H) = 10$, $h(T) = -10$) shows that $B_n = S_{n+1} - S_n$ is independent on $\mathcal{F}_n$. Thus, it follows from the independence lemma that

$$\mathbb{E}^{\mathbb{Q}}[f(S_{n+1})|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[f\left(S_n + h(X_{n+1})\right)|\mathcal{F}_n] = g(S_n), \tag{3}$$

where g is the function $g(s) := \mathbb{E}^{\mathbb{Q}}[f\left(s + h(X_{n+1})\right)]$.

Remark: Notice that we don't actually need to compute g explicitly to conclude that S is Markov. However, for applications (for example to pricing) one should compute g as explicitly as possible. In this exercise

$$g(s) = \left(f\left(s + 10\right) + f\left(s - 10\right)\right)/2. \tag{4}$$

(d) Since the coin tosses are independent we get that $\mathbb{Q}(X_{n+1} = H) = \mathbb{Q}(X_n = H | X(n) = \omega(n)) = \tilde{P}_n$. Since the $\tilde{P}_n$ are also constant in n, it follows that the $(X_n)_n$ are identically distributed

(e) Since under $\mathbb{Q}$ the $(X_n)_n$ are IID, it is easy to compute $\mathbb{Q}(\{\omega(n)\}) = p^{H_n(\omega(n))}(1-p)^{n-H_n(\omega(n))}$, where $p = \tilde{P}_n$. Since $p = 1/2 = 1 - p$, this expression simplifies further to $\mathbb{Q}(\omega(n)) = 1/2^n$ for all $\omega(n)$: any sequence $(\omega_1, \dots, \omega_n)$ of heads and tails has the same probability $\frac{1}{2^n}$ under $\mathbb{Q}$!

(f) Using the first method means evaluating $\mathbb{E}^{\mathbb{Q}}[(S_N - k)^+]$. To do this, one needs to generate a rv with the distribution which S_N has under $\mathbb{Q}$; let us try to identify it. If $B_n := S_n - S_{n-1}$ then $Y_n := \frac{1}{2}(\frac{B_n}{10} + 1)$ are Bernoulli IIDs. Thus $A_n := \sum_{k=1}^{n} Y_k$ has binomial distribution with parameters n and $p := \mathbb{Q}(Y_1 = 1) = \frac{1}{2}$, and $S_n - S_0 = 20A_n - 10n$. Since $S_0 = 80 = K$ we have that $C_N = (S_N - k)^+ = (20A_N - 10N)^+ = 10(2A_N - N)^+$. Since one can use a computer to quickly generate a binomial distribution,

$$C_0 = 10\mathbb{E}^{\mathbb{Q}}[(2A_N - N)^+], \qquad A_N \sim B\left(N, \frac{1}{2}\right). \tag{5}$$

can quickly be calculated on a computer. Besides, for big N one can approximate the binomial distribution with a Gaussian rv (by the central limit theorem), making the approximate calculation very fast.

The Markov method has a computational cost which increases with N^2 (as discussed in the lecture notes), and so a computer can use it to calculate C_0 even for big N.

(g) 1^{st} **solution:** While we could just use the formula eq. (5), let us instead do the calculations more 'by hand', which is probably clearer, and is closer to what one normally has to deal with (since often the law of S_N cannot be easily expressed using a well known distribution). We can compute

$$C_0 = \mathbb{E}^{\mathbb{Q}}[(S_5 - 80)^+] = \sum_{s \in \mathrm{Im}(S_5)} (s - 80)^+ \mathbb{Q}(\{\omega : S_5(\omega) = s\}), \tag{6}$$

where s belongs to the set $\mathrm{Im}(S_5)$ of values which S_5 can take. A moment's though (or just a drawing of the binomial tree of S) shows that $\mathrm{Im}(S_5) = \{130, 110, 90, 70, 50, 30\}$. Since each $\omega \in \{S_5 = s\}$ has $\mathbb{Q}$-probability $1/2^5$, the quantity of interest $\mathbb{Q}(\{\omega : S_5(\omega) = s\})$ equals $1/2^5$ multiplied times $\#\{S_5 = s\}$ (the number of elements of $\{S_5 = s\}$); let us compute it. In fact, to compute (6) we only need to do this for the s for which $(s - 80)^+ \neq 0$, since the other terms do not contribute to the sum in (6). There are exactly 3 such values of s (i.e. three ways for the call to 'expire in the money', i.e. to have a non-zero value at expiry): these are $130, 110, 90$. Clearly $S_5(\omega) = 130$ only if all five coin tosses are heads, so $\#\{S_5 = 130\} = 1$. $S_5(\omega) = 110$ iff exactly one of the five coin tosses is a tail; there are 5 sequences that have 4 heads and one tail (indeed these are:

$$THHHH, HTHHH, HHTHH, HHHTH, HHHHT),$$

and so $\#\{S_5(\omega) = 110\} = 5$. Finally, $S_5(\omega) = 90$ iff exactly two of the five coin tosses come out tail, so $\#\{S_5(\omega) = 90\}$ equals to the number of ways of choosing 2 tails in a sequence of 5 tosses. Recall that the number of ways to choose k tails out of n coin tosses is $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$; so $\#\{S_5(\omega) = 90\} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{5!}{2!3!} = 10$. In summary,

$$\mathbb{Q}(\{S_5 = 130\}) = 1/2^5, \mathbb{Q}(\{S_5 = 110\}) = 5/2^5, \mathbb{Q}(\{S_5 = 90\}) = 10/2^5\,,$$

and since $2^5 = 32$ the time-zero price of the call is

$$C_0 = \frac{1}{32}(130-80) + \frac{5}{32}(110-80) + \frac{10}{32}(90-80) = \frac{300}{32} = 9.375$$

2^{nd} **solution:** Alternatively, we can use the fact that S is Markov under $\mathbb{Q}$ and $C_N = f_N(S_N)$ for $f_N(s) := (s-K)^+$ to compute by backward induction the pricing functions $(f_n)_{n\le N}$ s.t. $C_n = \mathbb{E}^{\mathbb{Q}}[C_{n+1}|\mathcal{F}_n] = f_n(S_n)$.

In this exercise one can get nice explicit formulas for the $(f_n)_n$, since from eqs. (3) and (4) we get

$$f_n(s) = \left(f_{n+1}(s+10) + f_{n+1}(s-10)\right)/2. \tag{7}$$

from which, using $f_5(s) := (s-80)^+$, one can compute explicitly by backward induction:

$$\begin{aligned}
f_4(s) &= \tfrac{1}{2}\Big((s-70)^+ + (s-90)^+\Big)\\
f_3(s) &= \tfrac{1}{2^2}\Big((s-60)^+ + 2(s-80)^+ + (s-100)^+\Big)\\
f_2(s) &= \tfrac{1}{2^3}\Big((s-50)^+ + 3(s-70)^+ + 3(s-90)^+ + (s-110)^+\Big)\\
f_1(s) &= \tfrac{1}{2^4}\Big((s-40)^+ + 4(s-60)^+ + 6(s-80)^+ + 4(s-100)^+ + (s-120)^+\Big)\\
f_0(s) &= \tfrac{1}{2^5}\Big((s-30)^+ + 5(s-50)^+ + 10(s-70)^+ + 10(s-90)^+ + 5(s-110)^+ + (s-130)^+\Big)
\end{aligned}$$

In fact, looking at the above formulas for f_n is becomes immediately clear that, if $f_N(s) := (s-K)^+$ is the payoff function of a call with strike K and expiry $N \in \mathbb{N}$, then for $i = 0, \dots, N$ we find the call price at time $N-i$ is $f_{N-i}(S_{N-i})$, where

$$f_{N-i} = \frac{1}{2^i}\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix}\Big(s - K - 10i + 20k\Big)^+.$$

That this is the correct formula for f_{N-i} is a fact that, once correctly guessed, could be proved by induction, simply verifying that this formula satisfies eq. (7) and $f_N(s) := (s-K)^+$.

However, this is not normal: in most exercises the formulas for $(f_n)_n$ would be terribly ugly and have a very different form for different values of n. Thus, normally one does not compute a formula for $f_n(s)$ for every s, n. Rather, one computes the possible values $\{s_n^k\}_k$ of S_n, and then uses eq. (7) and the formula for f_N (in our case $f_5(s) := (s-80)^+$) to compute numerically $f_n(s_n^k)$ for all k. Let us do this explicitly. Since $f_5(s) := (s-80)^+$ and S_5 takes values $\{30, 50, 70, 90, 110, 130\}$, we can compute

S_5	30	50	70	90	110	130
$f_5(S_5)$	0	0	0	10	30	50

Since S_4 takes values $\{40, 60, 80, 100, 120\}$, eq. (7) with $n = 4$ and the above table for f_5 gives us

S_4	40	60	80	100	120
$f_4(S_4)$	0	0	$\frac{1}{2}(10)$	$\frac{1}{2}(30+10)$	$\frac{1}{2}(50+30)$

i.e.

S_4	40	60	80	100	120
$f_4(S_4)$	0	0	5	20	40

We can now compute f_3 using the fact that S_3 takes values $\{50, 70, 90, 110\}$, eq. (7) with $n = 3$ and the above table for f_4 to get

S_3	50	70	90	110
$f_3(S_3)$	0	$\frac{1}{2}(5+0)$	$\frac{1}{2}(20+5)$	$\frac{1}{2}(40+20)$

i.e.

S_3	50	70	90	110
$f_3(S_3)$	0	$\frac{5}{2}$	$\frac{25}{2}$	30

We can now compute f_2 using the fact that S_2 takes values $\{60, 80, 100\}$, eq. (7) with $n = 2$ and the above table for f_3 to get

S_2	60	80	100
$f_2(S_2)$	$\frac{1}{2}\left(\frac{5}{2}+0\right)$	$\frac{1}{2}\left(\frac{25}{2}+\frac{5}{2}\right)$	$\frac{1}{2}\left(30+\frac{25}{2}\right)$

i.e.

S_2	60	80	100
$f_2(S_2)$	$\frac{5}{4}$	$\frac{15}{2}$	$\frac{85}{4}$

We can now compute f_1 using the fact that S_1 takes values $\{70, 90\}$, eq. (7) with $n = 1$ and the above table for f_2 to get

S_1	70	90
$f_1(S_1)$	$\frac{1}{2}\left(\frac{15}{2}+\frac{5}{4}\right)$	$\frac{1}{2}\left(\frac{85}{4}+\frac{15}{2}\right)$

i.e.

S_1	70	90
$f_1(S_1)$	$\frac{35}{8}$	$\frac{115}{8}$

We can now finally compute f_0 using the fact that $S_0 = 80$, eq. (7) with $n = 0$ and the above table for f_1 to get

$$C_0 = f_0(S_0) = \frac{1}{2}\left(\frac{35}{8} + \frac{115}{8}\right) = \frac{150}{16}.$$

(h) As we computed in the previous item $C_0 = \frac{150}{16} = 9.375$

2. [default,M15]

Consider a market $(B_n, S_n)_{n=0,1,\dots,T}$ described by a multi-period binomial model with constant parameters $0 < d < 1 + r < u$, and as usual let $\mathcal{F}_k = \sigma(X_1, \dots, X_k), 0 \le k \le T$ be the filtration generated by the coin tosses $(X_i)_i$. Consider a *forward-start call option*, which entitles its holder to receive at time $T_0 \in \mathbb{N}, T_0 < T$ a call option (on the stock S) with maturity T and strike KS_{T_0} (where $K > 0$). Answer the following questions, and (other than in item (a)) justify carefully with proofs.

(a) Write down a formula, involving the expectation with respect to the risk-neutral measure $\mathbb{Q}$, for

$$V_0 := \text{the price at time 0 of the forward-start call option.}$$

(b) Show that, if $\{X_i\}_{i \in I}$ are independent random vectors and $\{f_i\}_{i \in I}$ are Borel functions then $\{f_i(X_i)\}_{i \in I}$ are independent random vectors

(c) Prove that the random variables

$$R_{k+1} := \frac{S_{k+1}}{S_k}, \quad k = 0, 1, \dots, T-1,$$

are IID under the EMM $\mathbb{Q}$.

(d) Prove that $\frac{S_T}{S_{T_0}}$ is independent of S_{T_0} under $\mathbb{Q}$.

(e) Compute the expectation of S_{T_0} under $\mathbb{Q}$.

(f) Show that $V_0 = c(T - T_0, Kx)$, where $c(t, x)$ is the price at time 0 of a call option with expiry t and $S_0 = x$.

Solution:

1. The risk neutral pricing formula gives that the price of the forward-start is

$$V_0 = \frac{1}{(1+r)^T}\mathbb{E}^{\mathbb{Q}}\left((S_T - KS_{T_0})^+\right).$$

2. To show that the random variables $Y_i := f_i(X_i), i \in I$ are independent, notice that for every Borel bounded functions $\{g_i\}_i$ we get that $g_i(Y_i) = (g_i \circ f_i)(X_i)$, and since the $\{X_i\}_{i\in I}$ are independent we get, for every finite set $J \subseteq I$,

$$\mathbb{E}^{\mathbb{Q}}[\Pi_{i\in J}(g_i \circ f_i)(X_i)] = \Pi_{i\in J}\mathbb{E}^{\mathbb{Q}}[(g_i \circ f_i)(X_i)]$$

i.e.

$$\mathbb{E}^{\mathbb{Q}}[\Pi_{i\in j}g_i(Y_i)] = \Pi_{i\in J}\mathbb{E}^{\mathbb{Q}}[g_i(Y_i)]$$

proving the thesis.

3. Let us show that, for each i, R_{i+1} is independent of $\mathcal{F}_i$; this implies that R_{i+1} is independent of $(R_1, \ldots, R_i)$ (since the $R_k, k \leq i$ are $\mathcal{F}_i$-measurable, i.e. $\sigma(R_1, \ldots, R_i) \subseteq \mathcal{F}_i$), and so that the $\{R_k\}_{k=0,\ldots,T-1}$ are independent (as stated in remark 99 in the lecture notes).

First, notice that X_{i+1} is $\mathbb{Q}$-independent of $(X_1, \ldots, X_i)$ since, by definition of $\mathbb{Q}$,

$$\mathbb{Q}(X_{i+1} = H|\omega_1, \ldots, \omega_i) = \tilde{p}_i := \frac{(1+r)-d}{u-d},$$

and the latter it deterministic (i.e. it does not depend on $\omega_1, \ldots, \omega_i$) in this exercise. Since R_{i+1} only depends on the $(i+1)^{th}$ coin toss X_{i+1} we can write $R_{i+1} = f_{i+1}(X_{i+1})$ for some (Borel) function f_{i+1}. By part b, $R_{i+1} = f_{i+1}(X_{i+1})$ is $\mathbb{Q}$-independent of $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$.

The $\{R_k\}_{k=0,\ldots,T-1}$ are also identically distributed, since $\{R_i = u\} = \{X_i = H\}$ has probability $\tilde{p}_i$, and $\tilde{p}_i$ does not actually depend on i.

4. Since the $\{R_k\}_{k=0,\ldots,T-1}$ are independent, the two vectors $(R_k)_{k=0,\ldots,T_0}$ and $(R_k)_{k=T_0+1,\ldots,T-1}$ are independent (as stated in remark 99 in the lecture notes). Since $\frac{S_T}{S_{T_0}} = R_{T_0+1}\cdots R_T$ and analogously $S_{T_0} = S_0R_1\cdots R_{T_0}$, $\frac{S_T}{S_{T_0}}$ is independent of S_{T_0} by part b.

5. Since the discounted stock price is a martingale under $\mathbb{Q}$,

$$\frac{1}{(1+r)^{T_0}}\mathbb{E}^{\mathbb{Q}}(S_{T_0}) = \frac{1}{(1+r)^0}\mathbb{E}^{\mathbb{Q}}(S_0) = S_0, \text{ and so } \mathbb{E}^{\mathbb{Q}}(S_{T_0}) = S_0(1+r)^{T_0}.$$

6. Since $S_{T_0} > 0$ we can write $(S_T - KS_{T_0})^+$ as the product of the two independent random variables S_{T_0} and $\left(\frac{S_T}{S_{T_0}} - K\right)^+$, and so

$$V_0 = \frac{1}{(1+r)^T}\mathbb{E}^{\mathbb{Q}}\left(S_{T_0}\right)\tilde{\mathbb{E}}\left[\left(\frac{S_T}{S_{T_0}} - K\right)^+\right],$$

which one could of course also have derived using the independence lemma. Since $\mathbb{E}^{\mathbb{Q}}\left(S_{T_0}\right) = x(1+r)^{T_0}$ and $x > 0$ we get that

$$V_0 = x\frac{1}{(1+r)^{T-T_0}}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{S_{T_0}} - K\right)^+\right] = \frac{1}{(1+r)^{T-T_0}}\mathbb{E}^{\mathbb{Q}}\left[\left(x\frac{S_T}{S_{T_0}} - Kx\right)^+\right].$$

Since the $(R_k)_k$ are IIDs, the random variables

$$x\frac{S_T}{S_{T_0}} = xR_{T_0+1}\cdots R_T, \quad S_{T-T_0} = S_0R_1\cdots R_{T-T_0}$$

have the same law (under $\mathbb{Q}$), and so

$$V_0 = \frac{1}{(1+r)^{T-T_0}}\mathbb{E}^{\mathbb{Q}}\left[(S_{T-T_0} - Kx)^+\right] = c(T - T_0, Kx).$$