This document contains 2 questions.

1. [default,M1]

Consider a binomial market model $(B_n, S_n)_{n\le N}$ where the bank account B has interest rate $r = 0$ and the price of the underlying S starts at $S_0 = 80$, and its value increases by 10 in case of Heads and decreases by 10 in case of Tails, i.e.

$$S_{n+1}(\omega) := \begin{cases} S_n(\omega_1, \dots, \omega_n) + 10, & \text{if } \omega_{n+1} = H \\ S_n(\omega_1, \dots, \omega_n) - 10, & \text{if } \omega_{n+1} = T \end{cases}, \qquad n \in 0, \dots, N-1.$$

Denote with $\mathbb{Q}$ the unique risk-neutral measure and with X_n the coin tosses, given as usual by $X_n(\omega) = \omega_n$.

(a) Draw the binary tree representing S. Can you draw it as a recombinant tree?

A. No B. Yes

(b) Are $(X_n)_{n\le N}$ independent under $\mathbb{Q}$?

A. No B. Yes

(c) Is S Markov under $\mathbb{Q}$?

A. No B. Yes

(d) Are $(X_n)_{n\le N}$ identically distributed under $\mathbb{Q}$?

A. No B. Yes

(e) Compute $\mathbb{Q}(\{\omega\})$ for every $\omega \in \{H,T\}^N$, then choose the correct statement

A. $\mathbb{Q}(\{\omega\})$ it not constant in N, nor in $\omega \in \{H,T\}^N$

B. $\mathbb{Q}(\{\omega\})$ it not constant in N, but is constant in $\omega \in \{H,T\}^N$

C. $\mathbb{Q}(\{\omega\}) = 1/2^N$ for all $N \ge 1, \omega \in \{H,T\}^N$

D. None of the above

(f) Consider the following methods to compute the price C_0 of a call option on S with strike $K = 80$ and maturity N.

1. Compute $\mathbb{Q}$, then use it to compute $C_0 = \mathbb{E}^{\mathbb{Q}}[C_N]$, where $C_N := (S_N - 80)^+$
2. Compute C_N , then use $C_n = \mathbb{E}^{\mathbb{Q}}[C_{n+1}|\mathcal{F}_n]$ to compute $(C_n)_{n=0}^{N-1}$ by backward induction

Which of the following statement (about computing C_0 numerically using a computer for big N, say $N > 100$) is correct?

A. Both methods allow to compute C_0 even for big N

B. Only the first method allows to compute C_0 even for big N

C. Only the second method allows to compute C_0 even for big N

D. Neither method allows to compute C_0 for big N

(g) Which of the following statement about computing C_0 by hand when $N = 5$ and using one of two methods above is correct?

A. Both methods allow to compute C_0 reasonably fast

B. Only the first method allows to compute C_0 reasonably fast

C. Only the second method allows to compute C_0 reasonably fast

D. Neither method allows to compute C_0 reasonably fast

(h) Compute C_0 by hand when $N = 5$, then choose the correct statement

A. $C_0 \in (6, 8)$ B. $C_0 \in [8, 10]$ C. $C_0 \in (10, 12)$ D. None of the above

2. [default,M15]

Consider a market $(B_n, S_n)_{n=0,1,\ldots,T}$ described by a multi-period binomial model with constant parameters $0 < d < 1 + r < u$, and as usual let $\mathcal{F}_k = \sigma(X_1, \ldots, X_k), 0 \le k \le T$ be the filtration generated by the coin tosses $(X_i)_i$. Consider a *forward-start call option*, which entitles its holder to receive at time $T_0 \in \mathbb{N}, T_0 < T$ a call option (on the stock S) with maturity T and strike KS_{T_0} (where $K > 0$). Answer the following questions, and (other than in item (a)) justify carefully with proofs.

(a) Write down a formula, involving the expectation with respect to the risk-neutral measure $\mathbb{Q}$, for

$$V_0 := \text{the price at time 0 of the forward-start call option.}$$

(b) Show that, if $\{X_i\}_{i \in I}$ are independent random vectors and $\{f_i\}_{i \in I}$ are Borel functions then $\{f_i(X_i)\}_{i \in I}$ are independent random vectors

(c) Prove that the random variables

$$R_{k+1} := \frac{S_{k+1}}{S_k}, \quad k = 0, 1, \ldots, T-1,$$

are IID under the EMM $\mathbb{Q}$.

(d) Prove that $\frac{S_T}{S_{T_0}}$ is independent of S_{T_0} under $\mathbb{Q}$.

(e) Compute the expectation of S_{T_0} under $\mathbb{Q}$.

(f) Show that $V_0 = c(T - T_0, Kx)$, where $c(t, x)$ is the price at time 0 of a call option with expiry t and $S_0 = x$.