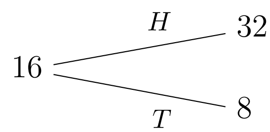
This document contains 2 questions.

1. [default,Q11]

A put option is the contract that gives the buyer the right (but not the obligation) to *sell* a share of the underlying to the seller at maturity at strike price $K > 0$. Consider the (arbitrage-free) one-period binomial model with interest rate $r = \frac{1}{4}$ and underlying S given by $S_0 = 16, S_1(H) = 32, S_1(T) = 8$, i.e. by the tree



If I have to hedge a short position in the put option with strike 16 (and maturity 1):

(a) What is the required initial capital V_0?

A. $\frac{1}{3}$ B. $-\frac{1}{3}$ **C.** $\frac{16}{5}$ D. $-\frac{16}{5}$ E. None of the above

(b) How much cash $k \in \mathbb{R}$ should I keep in the bank?

A. $\frac{128}{15}$ B. $-\frac{128}{15}$ C. $\frac{1}{3}$ D. $-\frac{1}{3}$ E. None of the above

(c) How many shares $h \in \mathbb{R}$ of the underlying should I buy?

A. $\frac{1}{6}$ B. $-\frac{1}{6}$ C. $\frac{1}{3}$ **D.** $-\frac{1}{3}$ E. None of the above

(d) What is the price of the put (i.e., at what price should the put be sold)?

A. V_0 B. $-V_0$ C. None of the above

Solution: First, one should notice that our market model is sensical, because it is arbitrage-free, since

$$u := S_1(H)/S_0 = 2, \quad d := S_1(T)/S_0 = 1/2, \quad 1 + r = 5/4$$

satisfy $d < 1 + r < u$.

Then, recall that to hedge a short position in a derivative is the same as replicating a derivative, so let us replicate the put.

1^{st} **solution:** As variables, let us use the position in the bank account k and in stocks h. We look for k, h which replicate the payoff $X_1 := (K - S_1)^+$ of the put, i.e. s.t.

$$V_1^{k,h}(\omega) := k(1 + r) + hS_1(\omega) = X_1(\omega) \quad \text{for both } \omega = H \text{ and } \omega = T,$$

i.e.

$$\begin{cases} k(1 + \frac{1}{4}) + h \cdot 32 &= (16 - 32)^+, \\ k(1 + \frac{1}{4}) + h \cdot 8 &= (16 - 8)^+, \end{cases}$$

which simplify to

$$\begin{cases} \frac{5}{4}k + 32h &= 0, \\ 24h &= -8, \end{cases}$$

and so

$$h = -\frac{1}{3}, \qquad k = -32h \cdot \frac{4}{5} = \frac{32 \cdot 4}{3 \cdot 5} = \frac{128}{15}.$$

Thus the initial capital of the replicating portfolio is

$$V_0 = k \cdot 1 + hS_0 = \frac{32 \cdot 4}{3 \cdot 5} - \frac{1}{3} \cdot 16 = \frac{4}{3 \cdot 5}(32 - 4 \cdot 5) = \frac{4 \cdot 12}{3 \cdot 5} = \frac{16}{5}.$$

Thus the price of the put is $X_0 = V_0 = \frac{16}{5}$, one should one keep in the bank $k = \frac{128}{15}$, and the hedging strategy is $h = -\frac{1}{3}$

2^{nd} **solution:** Alternatively, let us use as variables the trading strategy h and the initial capital $x = V_0$; these are connected to k, h via the relation $k = x - hS_0$. Moreover, let us work in discounted terms; recall that, given a process A_0, A_1, the discounted process $\overline{A}$ is given by $\overline{A}_0 := A_0, \overline{A}_1 := A_1/(1+r)$. We look for x, h which replicate the payoff $X_1 := (K - S_1)^+$ of the put, i.e. s.t.

$$\overline{V}_1^{x,h}(\omega) := x + h(\overline{S}_1(\omega) - \overline{S}_0) = \overline{X_1} \quad \text{for both } \omega = H \text{ and } \omega = T.$$

Since $\frac{1}{1+r} = \frac{4}{5}$, the above equation becomes

$$\begin{cases} x + h(\frac{4}{5} \cdot 32 - 16) & = \frac{4}{5}(16 - 32)^+, \\ x + h(\frac{4}{5} \cdot 8 - 16) & = \frac{4}{5}(16 - 8)^+. \end{cases}$$

Multiplying times $\frac{5}{4}$ and carrying out the summations and subtractions yields

$$\begin{cases} \frac{5}{4}x + 12h & = 0, \\ \frac{5}{4}x - 12h & = 8, \end{cases}$$

Adding and subtracting the equations yields respectively

$$\frac{2 \cdot 5}{4}x = 8, \quad 2 \cdot 12h = -8, \quad \text{and so } x = \frac{16}{5}, \quad h = -\frac{1}{3}.$$

Thus, at time 0 one should keep in the bank $k = x - hS_0 = \frac{16}{5} + \frac{1}{3} \cdot 16 = \frac{128}{15}$.

2. [default,O16]

Consider a one-period binomial model with expiry $T = 1$, of a market composed of one underlying S and a bank account with interest rate r. Recall that a call (resp. put) option with strike K is the contract that gives you the right (but not the obligation) to buy (resp. sell) one unit of the underlying at price K at maturity. Assume that $r = 0$, $S_0 = 8$, and S_1 can take the values 4 and 16. Given constants $k = 8, k_1 = 6, k_2 = 12, c = 4$, consider the following derivatives; when a derivative is only specified in words, you first have to write the formula for its payoffs.

(a) A *capped call*, which has payoff function

$$C(x) := \min\left[(x - k)^+, c\right],$$

i.e. the payoff of the derivative is $C(S_T)$ at expiry. What is the price of this derivative?

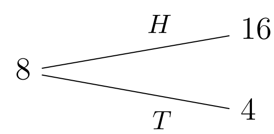
A. -4/3 B. 6 **C. 4/3** D. None of the above

(b) A (long) *range-forward* contract is the portfolio long 1 put of strike k_1, and short 1 call of strike $k_2 > k_1$; the call and put have the same underlying and expiry. What is the price of this derivative?

A. -2/3 B. 4 **C. 0** D. None of the above

Solution:

(a) We can represent S with a binomial tree as follows

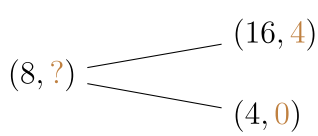


where we assume that the underlying probability space is $\{H, T\}$ and by convention the upper (resp. lower) branch of the tree corresponds to obtaining H=Heads (resp. T=Tails) when throwing a coin toss.

Since the *capped call* has payoff

$$\min\left[S_T - k, c\right](\omega) = \begin{cases} (16-8)^+ \wedge 4 = 4, & \omega = H \\ (4-8)^+ \wedge 4 = 0, & \omega = T \end{cases}$$

the (binomial) tree of (S, C) (i.e. representing (S, C)) is



Let us compute the price C_0 of the derivative by replication. We build a portfolio with initial capital x and a stock position h in the asset, and to replicate the option value we solve the equation

$$x(1+r) + h(S_T - S_0(1+r)) = \min\left[S_T - k, c\right]$$

between random variables, i.e. the following system of equations

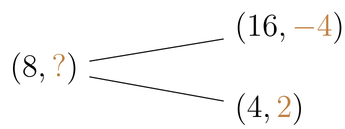
$$\begin{cases} x + 8h &= 4, \\ x - 4h &= 0, \end{cases} \tag{1}$$

whose solution is $h = 1/3, x = 4/3$.

(b) The payoff of the *range-forward* contract is

$$RF(S_T)(\omega) := (k_1 - S_T(\omega))^+ - (S_T(\omega) - k_2)^+ = \begin{cases} (6-16)^+ - (16-12)^+ = -4, & \omega = H \\ (6-4)^+ - (4-12)^+ = 2, & \omega = T \end{cases}$$

and so the tree of (S, RF) is



We build a portfolio with initial capital x and a stock position h in the asset, and to replicate the option value we solve the equation

$$x(1+r) + h(S_T - S_0(1+r)) = (k_1 - S_T)^+ - (S_T - k_2)^+$$

between random variables, i.e. the following system of equations

$$\begin{cases} x + 8h &= -4, \\ x - 4h &= 2, \end{cases} \tag{2}$$

whose solution is $h = -1/2, x = 0$.