This document contains 2 questions.

1. [default,O4]

Consider a one period market model composed of a bank account with constant interest rate $r = 2$ and a stock whose price at time 0 is $S_0 > 0$. Assume that the stock price S_1 at time one is given by $S_1 = 1 + N$, where N is a Poisson distributed random variable with parameter one.

(a) For what values of S_0 is the market free of arbitrage?

A. $S_0 < 1/3$ B. $S_0 \le 1/3$ **C. $S_0 > 1/3$** D. $S_0 \ge 1/3$ E. $S_0 > 1$ F. $S_0 \ge 1$

(b) When this market is free of arbitrage, is it complete?

A. No B. Yes C. It depends on the exact value of S_0

Solution: a) The market is free of arbitrage iff (meaning if and only if) $S_0 > 1/3$. This solution can be easily guessed by hoping that the NA condition $d < 1 + r < u$, which works for a model where S_1 takes finitely many values (of which $S_0 u/S_0 d$ is the max/min), also works in this setting in which S_1 takes the countably many values $1, 2, 3, \ldots$. Indeed in this case $d = 1, u = \infty$, and since

$$1 + r = 3$$

we would get that NA holds iff

$$\frac{1}{S_0} < 3 < \frac{\infty}{S_0},$$

i.e. $S_0 > 1/3$. Let us now see a more formal proof of this fact.

1st Solution: The final payoff of the portfolio with 0 initial wealth and which buys h stocks at time 0 (and to do so thus borrows hS_0 from the bank) is

$$X_1 = hS_1 - (1 + r)hS_0 = h(1 + N - 3S_0).$$

In particular if $h < 0$ then $X_1 \ge 0$ holds iff $N \le 3S_0 - 1$, and if $h > 0$ then $X_1 \ge 0$ holds iff $N \ge 3S_0 - 1$. Since $\{N \le 3S_0 - 1\}$ has never probability one (no matter the value of S_0), selling stocks can never lead to an arbitrage. Instead $\{N \ge 3S_0 - 1\}$ has probability one iff $S_0 \le 1/3$, and in this case $\{N > 3S_0 - 1\} \supseteq \{N > 0\}$, which has strictly positive probability, so buying any (strictly positive) amount of stocks (with money borrowed from the bank) leads to an arbitrage.

2nd Solution: Let us look for EMMs (equivalent martingale measures), i.e. $\mathbb{Q} \sim \mathbb{P}$ s.t. $S_0 = \mathbb{E}^{\mathbb{Q}}[S_1/(1 + r)]$. Let $q_n := \mathbb{Q}(N = n)$ for $n \in \mathbb{N}$, then $\mathbb{Q}$ is an EMM iff $q_n > 0$ for all n, $\sum_{n \in \mathbb{N}} q_n = 1$ and $3S_0 = \sum_{n \in \mathbb{N}} q_n(1 + n)$. Thus, if $\mathbb{Q}$ is an EMM then $q_0 \neq 1$ and so

$$\sum_{n \in \mathbb{N}} q_n(1 + n) > \sum_{n \in \mathbb{N}} q_n,$$

which gives $3S_0 > 1$ and so $S_0 > 1/3$. Conversely, if $S_0 > 1/3$ then one $3S_0 > 1$, and so it is intuitively clear that $3S_0$ can be written as a convex combination of $\{1 + n\}_{n \in \mathbb{N}}$ with strictly positive coefficients.

One can (and should) also prove this formally, but it is a bit tedious, so we skip the details but just give the idea. Choose a $m > 3S_0$, and $\epsilon > 0$ small enough that $m - \epsilon > 3S_0 > 1 + \epsilon$. Define $t_n^\epsilon := \epsilon/2^{n+1} > 0$, so that $\sum_{n\geq 0} t_n^\epsilon = \epsilon$. Define $q_n := t_n^\epsilon$ for all $n \in \mathbb{N} \setminus \{0, m\}$. Then there exists unique $q_0, q_m > 0$ such that $\sum_n q_n = 1$ and $3S_0 = \sum_{n\in\mathbb{N}} q_n(1 + n)$.

b) This market is not complete.

1st Solution: since the replication equality $X_1 = V_1$ (where $V_1 = k(1+r)+hS_1$) reads $X_1 = 3k+h(1+n)$ on $\{N = n\}$, and thus corresponds to a system of infinitely many independent equations (one for each $n \in \mathbb{N}$) in the two variables k, h, which has in general no solution.

For an explicit example of a payoff which is not replicable consider $X_1 = f(N)$ for an f such that $f(0) = 1, f(1) = 2, f(n) \neq n+1$ for some $n = \bar{n} \geq 3$: then $3k+h(1+N) = X_1$ reads $3k+h(1+n) = f(n)$ on $\{N = n\}$, and the system for $n = 0, n = 1$ has the unique solution $H^0 = 0, H^1 = 1$ which however does not work for $n = \bar{n}$.

2nd Solution: We look for EMMs. There are countable many variables $q_n, n \in \mathbb{N}$, which only have to satisfy two equations (and the countably many inequalities $q_n > 0$ for all n); so one would intuitively expect infinitely many solutions, and in particular several EMM, which would imply that the market is not complete. To give a full proof, we should prove that there are at least two EMM, say by constructing them explicitly. To do this, let $q_n = q_n(\epsilon, m)$ be as in part (a), and call $\mathbb{Q}^{\epsilon,m}$ be corresponding probability. Now choose a $m > 3S_0$, and let $\epsilon_1 \neq \epsilon_2$ be small enough so that $q_n(\epsilon_i, m)$ is defined for $i = 1, 2$. Then $\mathbb{Q}^{\epsilon_1,m}, \mathbb{Q}^{\epsilon_1,m}$ are EMM, and they differ since $q_n(\epsilon_1, m) \neq q_n(\epsilon_2, m)$ at least for any $n \in \mathbb{N} \setminus \{0, m\}$.

2. [default,O19]

Consider the one-period binomial model with interest rate $r = 0$ and stock price with value $S_0 = 150$ at time 0, and which takes the values $200, 140$ at time 1; thus, we model its price at time 1 with the random variable $S_1(H) = 200, S_1(T) = 140$, defined on $\Omega := \{H, T\}$. Assume that the probability of $\{S_1 = 200\}$ is $\frac{9}{10}$, i.e. $\mathbb{P}(\{H\}) = \frac{9}{10}$.

(a) Is this model arbitrage free ?

A. No **B. Yes**

(b) Consider from now on the call option with strike $K = 150$. What is the expected value $\mathbb{E}[C_1]$ of the call ?

A. 50 **B. 45** C. 44 D. 194

(c) What is the initial value of a portfolio which is replicating the call ? Try to solve the problem in two similar ways:

i. Describing the portfolio as (k, h), where k the number of bonds and h of shares, whose wealth is $V_t^{k,h} := kB_t + hS_t$ for both $t = 0$ and $t = 1$.

ii. Describing the portfolio as (x, h), where $x = k + S_0 h$ is the initial capital and h the number of shares, and whose wealth is $V_0^{x,h} := x$, $V_1^{x,h} := x(1 + r) + h(S_1 - S_0(1 + r))$.

A. The call is not replicable B. For multiple possible values **C. $\frac{50}{6}$** D. $-\frac{350}{3}$ E. $\frac{350}{3}$

(d) Suppose that we extend the binomial market, so that one can trade not only the bond and the stock, but also the call option at (initial) price $C_0 := p$. In the extended market (B, S, C), analogously to item (c) ii, describe a portfolio as (x, h, y), where x is the initial capital, h the number of shares, and y the number of options.

i. What random variable do I need to require to be always ≥ 0, and sometimes > 0, for the corresponding portfolio to be an arbitrage?

A. $h(S_1 - S_0(1 + r)) + y(C_1 - p(1 + r))$.

B. $h(S_1 - S_0(1 + r)) + y(C_1 - p)$.

C. $x(1 + r) + h(S_1 - S_0(1 + r)) + y(C_1 - p(1 + r))$.

D. $x(1 + r) + h(S_1 - S_0(1 + r)) + y(C_1 - p)$.

E. None of the above.

ii. If I *buy one* option, for which values of p can I make an arbitrage (by appropriately investing also in the bond and the stock) ?

A. None of the other answers B. For no value of p C. $p = \frac{50}{6}$ D. $p \neq \frac{50}{6}$ E. $p > \frac{50}{6}$ F. $p \geq \frac{50}{6}$ **G.** $p < \frac{50}{6}$ H. $p \leq \frac{50}{6}$

iii. If $p = 0$ and I *buy one* option, for what values of h do I get an arbitrage?

A. any $h \leq 0$ B. $h \in (0, 1)$ C. $h \in [0, 1]$ D. $h \in (-1, 0)$ **E.** $h \in [-1, 0]$

iv. If I *sell one* option, for which values of p can I make an arbitrage (by appropriately investing also in the bond and the stock) ?

A. None of the other answers B. For no value of p C. $p = \frac{50}{6}$ D. $p \neq \frac{50}{6}$ **E.** $p > \frac{50}{6}$ F. $p \geq \frac{50}{6}$ G. $p < \frac{50}{6}$ H. $p \leq \frac{50}{6}$

v. For which value(s) of $p \in \mathbb{R}$ is the extended market (B, S, C) arbitrage-free?

A. For $p = \mathbb{E}[C_1]$

B. For $p =$ the initial value of a portfolio which is replicating the call

C. For $p = \frac{350}{3}$

D. For $p = 50$

E. There is no such value of p

F. None of the above

Solution:

(a) The binomial model is arbitrage-free iff $d < 1 + r < u$. Since here $d = \frac{140}{150}, u = \frac{200}{150}$, this means that $r = 0$ must belong to $(d - 1, u - 1) = (-\frac{1}{15}, \frac{1}{3})$, which is does.

(b) $\mathbb{E}[C_1] = \mathbb{E}\big[(S_1 - K)^+\big]$ equals

$$0.9 \cdot (200 - 150)^+ + 0.1 \cdot (140 - 150)^+ = 0.9 \cdot 50 + 0.1 \cdot 0 = 45.$$

(c) i. The call replicable iff there exist $k, h \in \mathbb{R}$ such that $V_1^{k,h} := k(1 + r) + hS_1$ equals $(S_1 - K)^+$ a.s.. This equality between random variables leads to the system

$$\begin{cases} k + 200h = 50 \\ k + 140h = \ \ 0 \end{cases},$$

which has the unique solution $h = \frac{5}{6}, k = -\frac{350}{3}$, so the initial capital is $x = k + S_0 h = \frac{50}{6}$.

ii. The call replicable iff there exist $x, h \in \mathbb{R}$ such that $V_1^{x,h} := x + h(S_1 - S_0(1+r))$ equals $(S_1 - K)^+$ a.s.. This equality between random variables leads to the system

$$\begin{cases} x + 50h = 50 \\ x - 10h = \ \ 0 \end{cases},$$

which has the unique solution $h = \frac{5}{6}, x = \frac{50}{6}$.

(d) i. By definition, in an arbitrage the initial capital x equals 0. The money yp borrowed to buy $y \in \mathbb{R}$ options needs to be repayed back to the bank with interest.

ii. To find an arbitrage we need to ask

$$V_1 := V_1^{h,y}(p) := h(S_1 - S_0(1+r)) + y(C_1 - p(1+r))$$

be to ≥ 0 for both Heads and Tails, and then check if any of the solutions leads to a V_1 which is not $= 0$ in both cases. Buying (*resp. selling*) one option means taking $y = 1$ (*resp.* $y = -1$). This leads us to the system

$$\begin{cases} 50h + (50 - p)y \geq 0 \\ -10h + \quad (-p)y \geq 0 \end{cases},$$

which we rewrite as

$$\begin{cases} h \geq \quad (\frac{p}{50} - 1)y \\ h \leq - \qquad \frac{p}{10}y \end{cases}, \tag{1}$$

in the variable h, with $y = \pm 1$; here p is a fixed parameter (not a variable).
Now, if $y = 1$, the h which solve the system are $h \in [(\frac{p}{50} - 1), -\frac{p}{10}]$. If $(\frac{p}{50} - 1)y < -\frac{p}{10}y$ (i.e. $(p - 50) + 5p < 0$, i.e. $p < \frac{50}{6}$), every $h \in [(\frac{p}{50} - 1), -\frac{p}{10}]$ satisfies both inequalities in (1), and at least one of which with strict inequality (if $h = (\frac{p}{50} - 1)$ the second inequality is strict, if $h = -\frac{p}{10}$ the first one is). If instead $(\frac{p}{50} - 1) = -\frac{p}{10}$ (i.e. $p = \frac{50}{6}$) the interval reduces to just a point, so the only solution h satisfies both inequalities with equality, and so it is not an arbitrage. Thus, there is an arbitrage iff $p < \frac{50}{6}$, and then h is arbitrage iff $h \in [(\frac{p}{50} - 1), -\frac{p}{10}]$. Now a couple of comments:

Clearly the smaller the value of p the better it is to buy one option, so if for any value of $p = p_0$ there exists an arbitrage, then for any $p \leq p_0$ the same strategy is still an arbitrage. Thus, the solution must be of the form $(-\infty, a), (-\infty, a]$ or $\emptyset$; in fact, it turns out it is never $\emptyset$ (analogously the above reasoning shows that the only possible solutions if I was *selling* an option would be of the form $(a, \infty), [a, \infty)$ or $\emptyset$, though $\emptyset$ is not actually possible).

One can solve eq. (1) without fixing the value of y by thinking geometrically: each inequality represents a half-plane through the origin; drawing these two half-planes we can find their intersection. This intersection is empty or just a line if and only if for no values of h, y are the above inequalities both satisfied, with at least one inequality being satisfied strictly.

iii. As explained above I get an arbitrage iff $h \in [(\frac{p}{50} - 1), -\frac{p}{10}]$; taking $p = 0$ this means $h \in [-1, 0]$. Notice that $(-1, 0)$ was obviously a wrong answer, because $h = 0$ is obviously an arbitrage (you just get one option for free).

iv. Taking $y = -1$ in eq. (1) we can carry on the above computations will all the signs reversed, and we find that there exists an arbitrage h iff $p > \frac{50}{6}$.

(e) It trivially follows from definitions that, if $t > 0$, a portfolio H is an arbitrage iff tH is an arbitrage. We know that there is no arbitrage in the original market (bond+stock), so if there is an arbitrage strategy (x, h, y) in the extended market, it must have a non-zero amount y of options. If $y > 0$, we can take $t = \frac{1}{y}$, and we find that there is an arbitrage strategy which *buys one* call option, and trades $h^* := th \in \mathbb{R}$ shares.

If instead $y < 0$, we can take $t = -\frac{1}{y}$, and we find that there is an arbitrage strategy which *sells one* call option, and trades $h^* := th \in \mathbb{R}$ shares.

Thus, there exists an arbitrage in the extended market iff there is an arbitrage obtained by either buying exactly one call, or selling exactly one call, and investing appropriately also in the bond and the stock. Notice that this reasoning and conclusion works for any (linear) model (not just the binomial, not just in one-period), so this is worth remembering.

Since in the previous items we found that if $p < \frac{50}{6}$ (*resp.* $p > \frac{50}{6}$) we can make an arbitrage by buying (*resp. selling*) one option, the market is free of arbitrage if and only if neither $p < \frac{50}{6}$ nor $p > \frac{50}{6}$ is satisfied, i.e. iff $p = \frac{50}{6}$. This is equal to the replication price of the option, a fact that we will prove always to hold.