This document contains 1 questions.

1. [default,O28]Consider a stock traded in Europe, whose price (in €) at time $n = 0, 1$ is S_n, and suppose the exchange rate between £ and € (defined as the cost of one € in £) at time $n = 0, 1$ is E_n. Assume that the domestic (£) interest rate is $r_d = \frac{1}{9}$, the foreign (€) interest rate $r_f = \frac{1}{5}$, and E, S are as follows

$S_0 := 10, \quad E_0 := 1,$

ω	ω_1	ω_2	ω_3
$E_1(\omega)$	$\frac{5}{3}$	1	$\frac{1}{3}$
$S_1(\omega)$	$\frac{40}{3}$	$\frac{10}{9}$	$\frac{20}{3}$

(a) Is this market-model arbitrage-free? *(Hint: to determine what is the market, consider what investments you can make, and measure their values in £)*

A. No B. Yes

(b) Is this market-model complete?

A. No **B. Yes**

(c) Consider the call option on S with strike $K =$ €11. Is it replicable?

A. No **B. Yes**

(d) What is the set of arbitrage-free prices in domestic currency (£) of the above call option?

A. an open interval B. a singleton **C. $\emptyset$, i.e., there are no arbitrage-free prices**

(e) Suppose now that r_d was equal to 1/4, not to 1/9. Is the market arbitrage-free?

A. No **B. Yes**

Solution:

1. The first thing we need to understand is what exactly is the market in question. Since the value of the stock's price and foreign bank account measured in domestic currency are $X_n = S_n E_n$ and $Y_n = E_n(1 + r_f)^n, n = 0, 1$, the market we could consider is (B, X, Y), where $B_0 = 1, B_1 = 1 + r_d$ is the value of the domestic bond. We compute

$X_0 := 10, \quad Y_0 := 1,$

ω	ω_1	ω_2	ω_3
$X_1(\omega)$	$\frac{200}{9}$	$\frac{10}{9}$	$\frac{20}{9}$
$Y_1(\omega)$	2	$\frac{6}{5}$	$\frac{2}{5}$

1^{st} **solution:** By definition, the portfolio (h, g) is an arbitrage is the corresponding discounted final wealth is

$$V_1^{0,h,g} = h(X_1 - X_0(1 + r)) + g(Y_1 - Y_0(1 + r)),$$

satisfies $V_1^{0,h,g}(\omega) \geq 0$ for all ω, and not all inequalities are satisfies with equality. So, let us solve the system of inequalities $\frac{9}{10} V_1^{0,h,g} \geq 0$, i.e.

$$\begin{cases} h(20 - 10) + g\frac{9*5-25}{25} \geq 0 \\ h(1 - 10) + g\frac{9*3-25}{25} \geq 0 \\ h(2 - 10) + \quad g\frac{9-25}{25} \geq 0 \end{cases}$$

Carrying out the sum and subtractions and replacing g with $f := \frac{2g}{25}$ to get rid of fractions we find

$$\begin{cases} 10h + 10f \geq 0 \\ -9h + \quad f \geq 0 \\ -8h - \ \ 8f \geq 0 \end{cases},$$

Dividing the 1st inequality by 10 and the 3rd by 8, and isolating the variable f we find the system

$$\begin{cases} f \geq -h \\ f \geq 9h \\ f \leq -h \end{cases}.$$

Thus the 1st and 3rd inequalities together are equivalent to $f = -h$, and the 2nd inequality then becomes $-h \geq 9h$, i.e. $h \leq 0$. In this case if $h < 0$ the 2nd inequality is satisfied strictly, and so $h < 0$ and $f = -h$ (or equivalently $g = -\frac{25}{2}h$) is an arbitrage. So, there is arbitrage in this model.

2^{nd} **solution:** The risk-neutral probabilities $\mathbb{Q}$ are determined by

$$\begin{cases} \mathbb{E}^{\mathbb{Q}} X_1 & = & X_0(1 + r_d) \\ \mathbb{E}^{\mathbb{Q}} Y_1 & = & Y_0(1 + r_d) \\ \mathbb{Q}(\Omega) & = & 1 \\ \mathbb{Q} & \sim & \mathbb{P} \end{cases} \tag{1}$$

Since $1 + r_d = \frac{10}{9}$, multiplying the 1st equation times $9/10$ and the second times $5/2$ we find the system of equations

$$\begin{cases} 20q_1 + \ \ q_2 + 2q_3 = 10 \\ 5q_1 + 3q_2 + \ \ q_3 = \frac{25}{9} \\ q_1 + \ \ q_2 + \ \ q_3 = \ 1 \end{cases}$$

and on which we have to impose additionally the conditions $q_i > 0 \forall i$. Since the system has a unique solution, given by

$$q_1 = \frac{4}{9}, \quad q_2 = 0, \quad q_3 = \frac{5}{9}$$

and this does not satisfy $q_i > 0 \forall i$, there exists no EMM (there does exists one MM, but it is not equivalent to the physical measure $\mathbb{P}$). It follows that the model is NOT arbitrage-free.

2. Notice that we cannot apply the 2nd FTAP to conclude whether the model is complete or not, since the 2nd FTAP has as assumption the fact that there is no arbitrage. The model is complete (so in particular the call option is replicable), since we are in a trinomial model with three assets (the domestic bond B,X,Y), so the replication equation $h^1 B_1 + h^2 X_1 + h^3 Y_1 = D_1$ for the derivative with payoff D_1 is a system of 3 equations in the 3 unknowns h^1, h^2, h^3. Since the vectors representing $B_1 = (1 + r_d), X_1, Y_1$ are independent (as it can be easily checked, for example by showing that Y_1 cannot be replicated using B and X, or by showing that the determinant of the square matrix, whose columns are given by the values of B, X, Y, is not 0), so are the corresponding equations, so the system will have a (unique) solution.

3. The derivative can be shown to be replicable by solving the replication equation; in particular, notice that whether Y is replicable or not depends only on Y_1, not on Y_0. However, no computation is needed in this case: since the market is complete, we know that *every* derivative is replicable.

4. There is no price at which the call can be sold and s.t. the enlarged market is arbitrage-free: since the original market has an arbitrage, the enlarged market always has an arbitrage: for example, you can just choose the same strategy as in the original market, and do not trade in the additional derivative.

5. 1^{st} **solution:** Since $1 + r_d = \frac{5}{4}$, we find the system of inequalities $V_1^{0,h,g} \geq 0$ we find

$$\begin{cases} h(20 \cdot \frac{10}{9} - 10 \cdot \frac{5}{4}) + g(5 \cdot \frac{2}{5} - 1 \cdot \frac{5}{4}) \geq 0 \\ h(1 \cdot \frac{10}{9} - 10 \cdot \frac{5}{4}) + g(3 \cdot \frac{2}{5} - 1 \cdot \frac{5}{4}) \geq 0 \\ h(2 \cdot \frac{10}{9} - 10 \cdot \frac{5}{4}) + g(1 \cdot \frac{2}{5} - 1 \cdot \frac{5}{4}) \geq 0 \end{cases}$$

Replacing h, g with $h' := h\frac{1}{9 \cdot 4} = h/36, g' := g\frac{1}{5 \cdot 4} = g/20$ we compute

$$\begin{cases} h'(800 - 450) + g'(40 - 25) \geq 0 \\ h'(40 - 450) + g'(24 - 25) \geq 0 \\ h'(80 - 450) + g'(8 - 25) \geq 0 \end{cases}$$

Carrying out the algebra and isolating the g' terms gives

$$\begin{cases} g' \geq -\frac{350}{15}h' \\ g' \leq -410h' \\ g' \leq -\frac{370}{17}h' \end{cases},$$

from which we get

$$\begin{cases} -\frac{350}{15}h' \leq -410h' \\ -\frac{350}{15}h' \leq -\frac{370}{17}h' \end{cases},$$

or equivalently

$$\begin{cases} h \leq 0 \\ h \geq 0 \end{cases},$$

whose only solution is $h' = 0$, which implies $g' = 0$; this for such solution all inequalities are satisfied with equality, there is no arbitrage in this model.

2^{nd} **solution:** To look for EMMs, we solve the system of equations eq. (1) with $1 + r_d = \frac{5}{4}$, i.e.

$$\begin{cases} 20q_1 + q_2 + 2q_3 = \frac{45}{4} \\ 5q_1 + 3q_2 + q_3 = \frac{25}{8} \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

and on which we have to impose additionally the conditions $q_i > 0 \forall i$. The system has a unique solution, given by

$$q_1 = \frac{33}{64}, \quad q_2 = \frac{1}{32}, \quad q_3 = \frac{29}{64},$$

and this solution does satisfy $q_i > 0 \forall i$, so there exists a unique EMM. It follows that the model is arbitrage-free, and complete.