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1. [default,O7]

In this exercise we consider the general one-period linear market model of an arbitrage-free market (with no imperfections) i.e. assume that the final wealth of an investor is

$$V_0^{x,h} := x, \quad V_1^{x,h} := x(1+r) + h \cdot (S_1 - S_0(1+r)), \quad (x,h) \in \mathbb{R} \times \mathbb{R}^m, \tag{1}$$

where x represent the initial capital and h the number of units of the underlying S in the portfolio, and $r > -1$ the interest rate for investing in a bank account. We will assume that $m = 1$, i.e. there is only one risky asset S, which is assumed to be strictly positive, i.e. it has value $S_1 > 0$ at time $T := 1$ and value $S_0 > 0$ at time 0. We will consider all models in the above class, i.e. we do not specify the law of the random variable S_1. Denote by $C_0(K)$ a time-zero arbitrage-free price of a call option (on S) with strike price $K > 0$.

Consider the inequalities

A. $(S_0 - K)^+ \leq C_0(K)$ B. $S_0 - \frac{K}{1+r} \leq C_0(K)$ C. $\left(S_0 - \frac{K}{1+r}\right)^+ \leq C_0(K)$ D. $C_0(K) \leq S_0$ E. $C_0(K) \leq S_0 - K$

For each inequality, consider the following questions, and choose one of the following answers:

Questions:

- Does the inequality hold in every model (in the given class)?
- If so, does the inequality hold with strict inequality in every model (in the given class), or does it fail to be strict in at least some models?

Answers:

1. The inequality fails in some models
2. The inequality holds in all models, but not always strictly
3. The inequality holds strictly in all models

In summary, answer the following questions:

(a) What is the right answer for inequality A?
√ **1** ◯ 2 ◯ 3

(b) What is the right answer for inequality B?
◯ 1 √ **2** ◯ 3

(c) What is the right answer for inequality C?
◯ 1 √ **2** ◯ 3

(d) What is the right answer for inequality D?
◯ 1 ◯ 2 √ **3**

(e) What is the right answer for inequality E?
√ **1** ◯ 2 ◯ 3

Hint: Although to prove that an inequality is satisfied in every model (i.e. for every choice of r, S_0, S_1, K) you cannot just restrict to considering binomial models, to find counter-examples it is often enough to consider binomial (or trinomial) models.

Solution: Of course we have to solve this exercise using the *domination principle* to prove a positive result, and building a counter-example to prove a negative result.

A The inequality $(S_0 - K)^+ \leq C_0(K)$ can fail, as we will soon show. It is however interesting to notice that the inequality $(S_0 - K)^+ < C_0(K)$ holds across all models s.t. $r > 0$ and

- either $K \leq S_0$,
- $\mathbb{P}(K < S_1) > 0$,

since in these cases at least one of the inequalities $S_0 - K < C_0(K)$ and $0 < C_0(K)$ holds, as it follows from the strict domination principle and the inequalities

- $S_1 - K(1+r) < S_1 - K \leq (S_1 - K)^+$, which holds a.s. (actually for all ω),
- $0 \leq (S_1 - K)^+$ a.s., and $=$ does not hold a.s.,

the 1st (*resp. 2nd*) one of which holds under the 1st (*resp. 2nd*) assumption above.

So, for $(S_0 - K)^+ \leq C_0(K)$ to fail, we look for an *arbitrage-free* model (as simple as possible, so that we can carry on the computations) where the above conditions are not satisfied. Taking $r < 0$ this becomes easy: consider the binomial model with $S_0 = 2, u = 2 = 1/d, r = -1/4, K = 1$, which is arbitrage-free since it satisfies $1/2 = d < 1 + r = 3/4 < u = 2$. In it, calculate explicitly $C_0(K)$ by solving $x(1+r) + h(S_1 - S_0(1+r)) = (S_1 - K)^+$, i.e.

$$\begin{cases} x(1-1/4) + h(4 - 2\cdot 3/4) = (4-1)^+ \\ x(1-1/4) + h(1 - 2\cdot 3/4) = (1-1)^+ \end{cases},$$

or equivalently

$$\begin{cases} 3x/4 + 5h/2 = 3 \\ 3x/4 - h/2 = 0 \end{cases},$$

whose solution is $h = 1, x = 2/3$, and so $C_0(K) = x = 2/3 < 1 = 2 - 1 = S_0 - K$.

B,C To prove inequality C

$$\max\left(0, S_0 - \frac{K}{1+r}\right) \leq C_0(K), \tag{2}$$

notice that $0 \leq (S_1 - K)^+$ implies that $0 \leq C_0(K)$, whereas $S_1 - K \leq (S_1 - K)^+$ implies inequality B

$$S_0 - \frac{K}{1+r} \leq C_0(K), \tag{3}$$

as it follows by the domination property, first applied to the call versus the zero portfolio, and then to the call versus the portfolio which at time 0 is long one stock and short $K/(1+r)$ cash. Thus, inequalities B,C hold in all models.

These inequalities are not necessarily strict. It is easy to see why: since for some model the inequality $K \leq S_1$ holds a.s., so $S_1 - K \leq (S_1 - K)^+$ holds with equality, and so also does (3) (by the replication property); since (2) holds, the fact that (3) holds with equality shows that (2) also does. It still remains to show that there actually exists one such model (i.e. s.t. $K \leq S_1$ a.s.), by providing an example of it. It is important to point out that the model in question needs to be arbitrage-free (this was of course among the assuptions in our question). So, for example choose the binomial model with $S_0 = 2, u = 2 = 1/d, r = 0$ (which is arbitrage-free since it satisfies $1/2 = d < 1 + r = 1 < u = 2$), and take $K = 1$, so that $S_1(H) = 4 \geq 1 = K$ and $S_1(T) = 1 \geq 1 = K$, i.e. $K \leq S_1$ holds a.s..

D The inequality $C_0(K) < S_0$ holds in every model, as it follows from the strict domination principle and the fact that at time 1 the payoffs of a call with strike K is $(S_1 - K)^+$, which is strictly smaller than the payoff S_1 of a stock, since if $S_1 - K \leq 0$ then $(S_1 - K)^+ = 0 < S_1$, whereas if $S_1 - K > 0$ then $(S_1 - K)^+ = S_1 - K < S_1$ (recall that we assumed $S_1, K > 0$).

E The inequality $C_0(K) \leq S_0 - K$ can fail. The idea is that if $r > 0$ then

$$S_1 - (1+r)K < S_1 - K \leq (S_1 - K)^+,$$

and so by the strict domination principle $S_0 - K < C_0(K)$. Choosing one such complete (and arbitrage-free) model, for example taking the binomial model with $S_0 = 2, u = 2 = 1/d, r = 1/2, K = 1$ (which is arbitrage-free since it satisfies $1/2 = d < 1 + r = 3/2 < u = 2$), then gives an example where $S_0 - K < C_0(K)$ holds (by the above reasoning), and moreover can easily double-check the answer by calculating explicitly $C_0(K)$ by solving

$$x(1+r) + h(S_1 - S_0(1+r)) = (S_1 - K)^+,$$

i.e.

$$\begin{cases} 3x/2 + h(4 - 2 \cdot 3/2) = (4-1)^+ \\ 3x/2 + h(1 - 2 \cdot 3/2) = (1-1)^+ \end{cases},$$

or equivalently

$$\begin{cases} 3x/2 + \ \ h = 3 \\ 3x/2 - 2h = 0 \end{cases},$$

whose solution is $h = 1, x = 4/3$, and so $C_0(K) = x = 4/3 > 1 = 2 - 1 = S_0 - K$.