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Question 1

(Total: 20 marks)

[default,M16]

Consider a market $(B_n, S_n)_{n=0,1,\dots,N}$ where the bank account B has constant interest rate r, and the price of the stock S starts at $S_0 = 5$, and its value increases from time n to time $n+1$ by $n+1$ in case of Heads and decreases by $n+1$ in case of Tails, i.e.

$$S_{n+1}(\omega) := \begin{cases} S_n(\omega_1, \dots, \omega_n) + (n+1), & \text{if } \omega_{n+1} = H \\ S_n(\omega_1, \dots, \omega_n) - (n+1), & \text{if } \omega_{n+1} = T \end{cases}, \qquad n \in 0, \dots, N-1.$$

As usual $(X_n)_n$ denotes the process of coin tosses X which generates $(S_i)_i$, and we take as filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n), 0 \le n \le N$, the natural filtration of X. Consider a *call option C with strike K and expiry τ*, where τ is a *random* time, i.e., the derivative which gives a payoff $C := (S_\tau - K)^+$ at time τ (S_τ is the random variable which takes the value S_n on the event $\{\tau = n\}$). Assume that $r = 0, N = 2, K = 2$ and τ is as follows

ω	HH	HT	TH	TT
$\tau(\omega)$	2	2	1	1

Answer the following questions, and justify carefully with proofs.

(a) (2 points) Prove that the above model $(B_n, S_n)_{n=0,1,\dots,N}$ is free of arbitrage.

(b) (2 points) Prove that τ is a stopping time, i.e., that $\{\tau = n\} \in \mathcal{F}_n$ for all $n = 0, \dots, N$. Prove that a random time σ is a stopping time if and only if $\{\sigma \le n\} \in \mathcal{F}_n$ for all n.

(c) (4 points) Prove that the call option C can be written as a sum of derivatives, each with a payoff at a deterministic (i.e. non-random) time. Determine these derivatives explicitly.

(d) (6 points) Determine the replicating strategy H and the arbitrage-free price V of C, in one of the following two ways: either using the decomposition you found in the previous item, or working directly by replication only up to time τ.

(e) (6 points) Suppose it becomes possible to trade C at price C_n at time $n = 0, \dots, N$, where

$$C_0 = 2, \quad C_1(H) = 4, \quad C_1(T) = 2.$$

Construct an arbitrage in the (B, S, C) market, and compute its final payoff.

Solution:

(a) The binomial model has no arbitrage iff $d_n < 1 + r_n < u_n$ for all $n \le N-1$. This property is verified for $N = 2$, since $S_0, S_1, S_2 > 0$ and so

$$d_n := \frac{S_n - (n+1)}{S_n} < 1 = 1 + r < \frac{S_n + (n+1)}{S_n} =: u_n, \quad n = 0, 1.$$

One can do this also by explicitly computing the values

$$u_1(H) = 8/6, \quad d_1(H) = 4/6, \quad u_1(T) = 6/4, \quad d_1(T) = 2/4, \quad u_0 = 6/5, \quad d_0 = 4/5.$$

Another way is to notice that $\tilde{p}_n \in (0,1)$, where

$$\tilde{p}_n := \frac{(1+r_n)-d_n}{u_n-d_n} = \frac{(1+r_n)S_n-(S_n-(n+1))}{2(n+1)} = \frac{n+1}{2(n+1)} = \frac{1}{2}.$$

Equivalently, the unique probability $\mathbb{Q}$, for which the coin tosses are IID with probability of Heads being 1/2 (and which corresponds to the transition probabilities $\tilde{p}_n$ calculated above), is an EMM, and so by the 1st FTAP the model has no arbitrage.

(b) Since $\{\tau \le 0\} = \emptyset$, $\{\tau \le 2\} = \{HH, HT, TH, TT\} = \{H,T\}^2 = \Omega$ we trivially have $\{\tau \le n\} \in \mathcal{F}_n$ for $n = 0, 2$, and since $\{\tau \le 1\} = \{TH, TT\} = \{X_1 = T\}$ we have $\{\tau \le 1\} \in \mathcal{F}_1 = \sigma(X_1)$.

The identities

$$\{\sigma \le n\} = \cup_{k=0}^n \{\sigma = k\}, \quad \{\sigma \le n\} \setminus \{\sigma \le n-1\} = \{\sigma = n\}$$

show that $\{\sigma = n\} \in \mathcal{F}_n$ for all n iff $\{\sigma \le n\} \in \mathcal{F}_n$ for all n.

(c) By definition $S_\tau = S_n$ on $\{\tau = n\}$, and so $C = (S_n - K)^+$ on $\{\tau = n\}$. In other words,

$$C = \textstyle\sum_{n=0}^N (S_n - K)^+ 1_{\{\tau=n\}} = \sum_{n=0}^N D_n^n, \quad \text{where } D_n^n := (S_n - K)^+ 1_{\{\tau=n\}} \tag{1}$$

is the payoff of a derivative D^n with expiry n; this payoff at time 0 is zero on the event $\{\tau \neq n\}$, and it equals $(S_n - K)^+$ otherwise. We will denote with D_k^n the value at time $k \le n$ of the derivative D^n, and with H_k^n the number of shares one should hold at time k to replicate D^n. Notice that the value of D_k^n is defined only for $k \le n$, since the derivative has expiry n; analogously, H_k^n is defined only for $k \le n-1$.

If $\{\tau = n\}$, then D^n has the same payoff D_n^n as the call option with strike K and expiry n, otherwise D^n has no payoff. Notice that D_n^n is $\mathcal{F}_n$-measurable (as it should be, to be the payoff of a derivative with expiry n), thanks to the fact that τ is a stopping time and thus

$$\{\tau = n\} = \{\tau \le n\} \setminus \{\tau \le n-1\} \quad \text{is } \mathcal{F}_n\text{-measurable.}$$

(d) We show two possible solutions; while the first one is conceptually simpler, the second one is more elegant and quicker.

1^{st} **solution:** Since each derivative with a deterministic expiry is replicable in the binomial model, the previous item shows that C is replicable, and shows also how we could hedge it and price it.

Here are the trees for D^1, D^2 (notice that $D^0 = 0$ in this exercise, because $\{\tau = 0\} = \emptyset$).

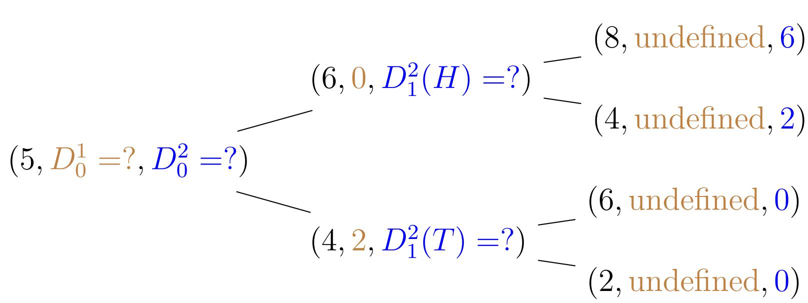

Figure 1: Tree of (S, D^1, D^2)

One can then proceed as usual by backward induction to price and hedge D^1, D^2, and then add the results to find the price and hedge of C. We can easily replicate D^1 by solving the system

$$\begin{cases} x + h(6-5) = 0 \\ x + h(4-5) = 2 \end{cases},$$

whose solution is easily found to be $x = 1, h = -1$, and so

$$D_0^1 = 1, H_0^1 = -1.$$

Analogously we find the values of $D_1^2(H), D_1^2(T), D_0^2$ by replication: the solution of

$$\begin{cases} x + h(8-6) = 6 \\ x + h(4-6) = 2 \end{cases}$$

is $x = 4, h = 1$, and so

$$D_1^2(H) = 4, H_1^2(H) = 1;$$

analogously the solution of

$$\begin{cases} x + h(6-4) = 0 \\ x + h(2-4) = 0 \end{cases}$$

is $x = 0, h = 0$, and so

$$D_1^2(T) = 0, H_1^2(T) = 0,$$

and finally to compute D_0^2 by backward induction find the solution of

$$\begin{cases} x + h(6-5) = 4 \\ x + h(4-5) = 0 \end{cases}$$

to be $x = 2, h = 2$, and so

$$D_0^2 = 2, H_0^2 = 2.$$

To compute V, notice that while D_2^1 is undefined, on the set $\{\tau = 2\}$ the quantity V_2 is defined and equals D_2^2. This is because, at time k, the value V_k of the derivative equals the present value of the *future* cash flows coming from the derivative, and so

$$V_k = \sum_{n=k}^{N} D_k^n, \qquad \text{on } \{k \leq \tau\}. \tag{2}$$

It follows that $V_1 = D_1^1 + D_1^2$ and $V_0 = D_0^1 + D_0^2$. Using the above calculations of D_k^n we thus find

$$V_1(H) = 0 + 4 = 4, \quad V_1(T) = 2 + 0 = 2, \quad V_0 = 1 + 2 = 3.$$

Analogously, to replicate the derivative we only need to replicate the cash flows which are in the future, and so

$$H_k = \sum_{n=k}^{N-1} H_k^n, \qquad \text{on } \{k \leq \tau - 1\}. \tag{3}$$

It follows that $H_1 = H_1^2$ and $H_0 = H_0^1 + H_0^2$ are given by

$$H_1(H) = 1, \quad H_1(T) = 0, \quad H_0 = -1 + 2 = 1.$$

2^{nd} **solution:** Let us work directly with C, without decomposing it. To do so, we draw below the binomial tree of the process (S, V). Notice that we know the value $V_\tau = C$ at time τ, and we have to determine the values of V_k at previous times, i.e. on $\{k \leq \tau\}$. In other words, we know

$$V_2 = (S_2 - K)^+ \quad \text{on } \{\tau = 2\} = \{HH, HT\}, \quad V_1 = (S_1 - K)^+ \quad \text{on } \{\tau = 1\} = \{TH, TT\}.$$

The value of V_k on $\{k > \tau\}$ is not defined (because the value at time $n > k$, of receiving an amount of cash V_k at time k, depends on how we choose to invest at times $n \geq k$).

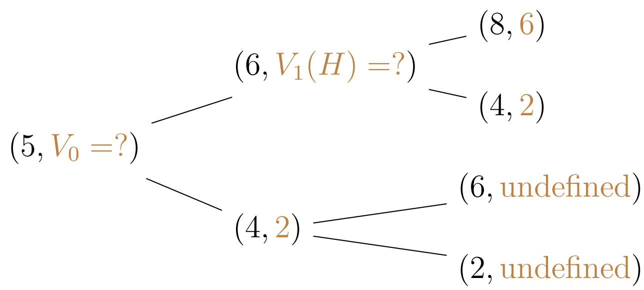

Figure 2: Tree of (S, V)

We can then determine the replicating strategy H and the price V as usual, by backward induction, either by hand (i.e., by solving the replication equations), or by using the delta-hedging formula and the RNPF. Either way, this approach is quicker, because it requires fewer calculations: while in our first solution method we had to compute the prices $D_0^1, D_0^2, D_1^2(H), D_1^2(T)$ and strategies $H_0^1, H_0^2, H_1^2(H), H_1^2(T)$, now we only need to compute the prices $V_0, V_1(H)$ and strategies $H_1(H), H_0$.

Let us now determine $H_1(H), V_1(H)$. If we do it by hand, we have to solve the equation

$$V_1(H) + H_1(H)\Big(S_2(H\omega_2) - S_1(H)(1+r)\Big) = V_2(H\omega_2), \quad \omega_2 \in \{H, T\},$$

between random variables, which corresponds to the system of equations

$$\begin{cases} x + h(8-6) = 6 \\ x + h(4-6) = 2 \end{cases},$$

in the variables $x := V_1(H), h := H_1(H)$, whose solution is easily found to be $x = 4, h = 1$. If we instead do it applying the delta-hedging and risk-neutral pricing formulas we get

$$H_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{6-2}{8-4} = 1$$

$$\tilde{p}_1(H) = \frac{(1 + r_1(H)) - d_1(H)}{u_1(H) - d_1(H)} = \frac{1 - \frac{4}{6}}{\frac{8}{6} - \frac{4}{6}} = \frac{1}{2}$$

$$V_1(H) = \tilde{p}_1(H)V_2(HH) + (1 - \tilde{p}_1(H))V_2(HT) = \frac{1}{2} \cdot 6 + (1 - \frac{1}{2}) \cdot 2 = 4.$$

We can now compute H_0, V_0. Let us only do it by hand, by solving the replication equation

$$V_0 + H_0\Big(S_1(\omega_1) - S_0(1+r)\Big) = V_1(\omega_1), \quad \omega_1 \in \{H, T\},$$

i.e., the system

$$\begin{cases} x + h(6-5) = 4 \\ x + h(4-5) = 2 \end{cases} .$$

in the variables $x := V_0, h := H_0$. Its solution is easily found to be $x = 3, k = 1$. In summary

$$V_1(H) = 4, \quad H_1(H) = 1, \quad V_0 = 3, \quad H_0 = 1.$$

(e) We know that $V_0 = 3$ is the unique value at time 0 of an arbitrage-free price process for C in the (B, S)-market. Thus, if C is traded at the prices listed in item (e), then there exists an arbitrage. Since $2 < 3$, it means the derivative is being traded at a smaller value than it should be, so to make an arbitrage we can: start with zero initial capital, *buy* the derivative, hedge such a trade (i.e. replicate minus the derivative, by trading in the stock market and using the bank), and put the remaining money in the bank. Since the derivative costs 2, and replicating it costs 3, we can put $3 - 2 = 1$ in the bank at time 0, and at the final time we will have $1(1+r)^N = 1$ in the bank.

Of course, there are many other possible arbitrage strategies. For example, one could simply borrow 2 from the bank and use that to buy one option at time 0. At time 1, if the option gives a payoff deposit that in the bank. If you do nothing else, your final wealth will be 2 in the case of HH, and 0 otherwise, so this strategy was also an arbitrage.