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Question 1

(Total: 20 marks)

[default,M16]

Consider a market $(B_n, S_n)_{n=0,1,\ldots,N}$ where the bank account B has constant interest rate r, and the price of the stock S starts at $S_0 = 5$, and its value increases from time n to time $n+1$ by $n+1$ in case of Heads and decreases by $n+1$ in case of Tails, i.e.

$$S_{n+1}(\omega) := \begin{cases} S_n(\omega_1, \ldots, \omega_n) + (n+1), & \text{if } \omega_{n+1} = H \\ S_n(\omega_1, \ldots, \omega_n) - (n+1), & \text{if } \omega_{n+1} = T \end{cases}, \qquad n \in 0, \ldots, N-1.$$

As usual $(X_n)_n$ denotes the process of coin tosses X which generates $(S_i)_i$, and we take as filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n), 0 \le n \le N$, the natural filtration of X. Consider a *call option C with strike K and expiry τ*, where τ is a *random* time, i.e., the derivative which gives a payoff $C := (S_\tau - K)^+$ at time τ (S_τ is the random variable which takes the value S_n on the event $\{\tau = n\}$). Assume that $r = 0, N = 2, K = 2$ and τ is as follows

ω	HH	HT	TH	TT
$\tau(\omega)$	2	2	1	1

Answer the following questions, and justify carefully with proofs.

(a) (2 points) Prove that the above model $(B_n, S_n)_{n=0,1,\ldots,N}$ is free of arbitrage.

(b) (2 points) Prove that τ is a stopping time, i.e., that $\{\tau = n\} \in \mathcal{F}_n$ for all $n = 0, \ldots, N$. Prove that a random time σ is a stopping time if and only if $\{\sigma \le n\} \in \mathcal{F}_n$ for all n.

(c) (4 points) Prove that the call option C can be written as a sum of derivatives, each with a payoff at a deterministic (i.e. non-random) time. Determine these derivatives explicitly.

(d) (6 points) Determine the replicating strategy H and the arbitrage-free price V of C, in one of the following two ways: either using the decomposition you found in the previous item, or working directly by replication only up to time τ.

(e) (6 points) Suppose it becomes possible to trade C at price C_n at time $n = 0, \ldots, N$, where

$$C_0 = 2, \quad C_1(H) = 4, \quad C_1(T) = 2.$$

Construct an arbitrage in the (B, S, C) market, and compute its final payoff.