This document contains 2 questions.

1. [default,Q3]

   Given X, Y independent, $X, Y \sim Unif[0,1]$, use the properties of the conditional expectation to compute $\mathbb{E}((X+Y)^2|X)$.

Solution: $\mathbb{E}((X+Y)^2|X) = X^2 + X + 1/3$, since $(X+Y)^2 = X^2 + 2XY + Y^2$ and

$$\mathbb{E}[X^2|X] = X^2, \quad \mathbb{E}[2XY|X] = 2X\mathbb{E}[Y] = X, \quad \mathbb{E}[Y^2|X] = \mathbb{E}[Y^2] = 1/3,$$

by linearity, and where we used respectively

1. Take out what is know
2. Take out what is know, and Independence
3. Independence

and using the density $1_{[0,1]}$ of $Y \sim Unif[0,1]$ of we computed

$$\mathbb{E}[Y] = \int_0^1 y dy = \frac{1^2}{2} - 0 = \frac{1}{2}, \qquad \mathbb{E}[Y^2] = \int_0^1 y^2 dy = \frac{1^3}{3} - 0 = \frac{1}{3}.$$

2. [default,P7]

   Which of the following statements about the process $Y = (Y_n)_{n\in\mathbb{N}}$ are correct?

   We are assuming that $\mathbb{E}|Y_n| < \infty$ for all $n \in \mathbb{N}$. Justify carefully with either proofs or counterexamples. Recall that a random variable X is said to be bounded if $|X| \leq c$ for some constant $c < \infty$.

   (a) If Y is a martingale, then $\mathbb{E}(Y_n) = \mathbb{E}(Y_0)$ for every n.

   (b) If Y is a martingale and τ a bounded stopping time then $\mathbb{E}(Y_\tau) = \mathbb{E}(Y_0)$.

   (c) If Y satisfies $\mathbb{E}(Y_n) = \mathbb{E}(Y_0)$ for every n then Y is a martingale.

   (d) If Y satisfies $\mathbb{E}(Y_\tau) = \mathbb{E}(Y_0)$ for every bounded stopping time τ then Y is a martingale.
   Hint: Given $s < t$, $A \in \mathcal{F}_s$, $\tau := t1_{A^c} + s1_A$, compare $\mathbb{E}(Y_\tau)$ with $\mathbb{E}(Y_t)$.

Solution:

(a) By the martingale property $Y_0 = \mathbb{E}_0(Y_n)$ and taking expectations and using the tower property of the conditional expectation we get $\mathbb{E}[Y_n] = \mathbb{E}[\mathbb{E}_0[Y_n]] = \mathbb{E}[Y_0]$.

(b) If $\tau \leq N \in \mathbb{N}$ then, since τ is a stopping time, by Kolmogorov's characterisation of the conditional expectation of $\mathbb{E}(Y_N|\mathcal{F}_n) = Y_n$ we get $\mathbb{E}[Y_N 1_{\{\tau=n\}}] = \mathbb{E}[Y_n 1_{\{\tau=n\}}]$. Thus, since by definition of Y_τ we have $Y_\tau 1_{\{\tau=n\}} = Y_n 1_{\{\tau=n\}}$, we have that

$$\mathbb{E}[Y_N] = \sum_{n=0}^{N} \mathbb{E}[Y_N 1_{\{\tau=n\}}] = \sum_{n=0}^{N} \mathbb{E}[Y_\tau 1_{\{\tau=n\}}] = \mathbb{E}[Y_\tau].$$

Since by item (a) we have $\mathbb{E}[Y_N] = \mathbb{E}[Y_0]$, we conclude $\mathbb{E}[Y_\tau] = \mathbb{E}[Y_0]$.

(c) In a binomial model where coin tosses are independent and heads and tails have the same probability, consider the process Z described by the following binomial tree:

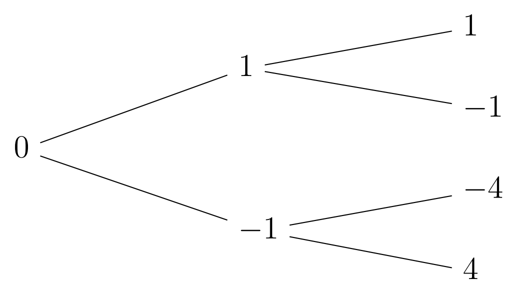

Figure 1: Tree of Z.

Since

$$\mathbb{E}_1[Z_2](T) = \frac{1}{2} \cdot (-4) + \frac{1}{2} \cdot 4 = 0 \neq -1 = Z_1(T),$$

Z is not a martingale, yet clearly $\mathbb{E}[Z_2] = \mathbb{E}[Z_1] = Z_0 = 0$.

(d) If $s < t$ and $A \in \mathcal{F}_s$ then $\tau := t 1_{A^c} + s 1_A$ is a stopping time, and so

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_\tau\right] = \mathbb{E}\left[X_t 1_{A^c}\right] + \mathbb{E}\left[X_s 1_A\right]$$

On the other hand, the constant t itself is a bounded stopping time, and so

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_t 1_{A^c}\right] + \mathbb{E}\left[X_t 1_A\right],$$

and comparing the two gives $\mathbb{E}\left[X_s 1_A\right] = \mathbb{E}\left[X_t 1_A\right]$, and so $\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s$, i.e. X is a martingale.