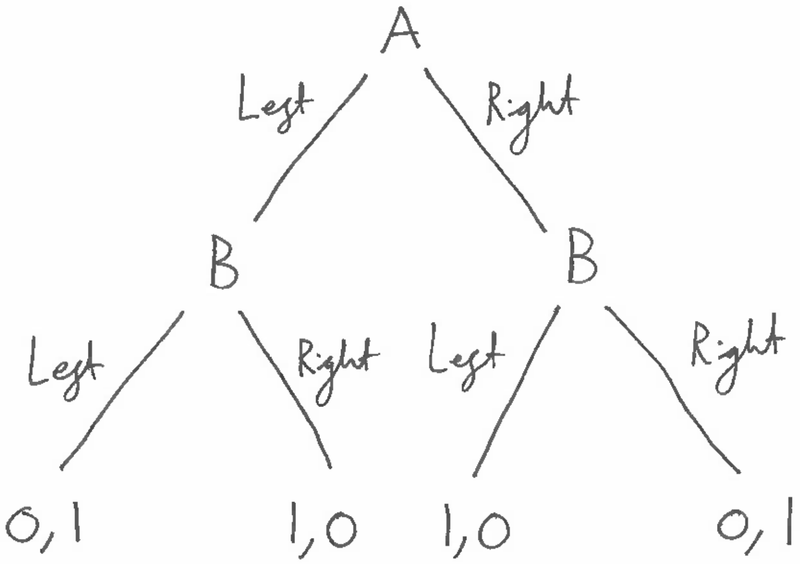
Intro to Game Theory: Problem Set 1 Solutions

1).

a). Let's assign a payoff of 1 to a player receiving a sweet and a payoff of zero otherwise. (Any value x for receiving a sweet and value y for no sweet where $x > y$ suffices, but 27.5 and -214 would seem unsensible!)

b).

(i).



(ii).

		B: Left	B: Right
A	Left	0, 1	1, 0
A	Right	1, 0	0, 1

(c). There is no optimal pure strategy for either player in this game. Notice that if player A chooses Left, player B should choose Left, but if player B chooses left, player A should chose right, but then player B should choose right. This continues around forming a cycle; ie. there is no optimal pure strategy for either player here.

(2)

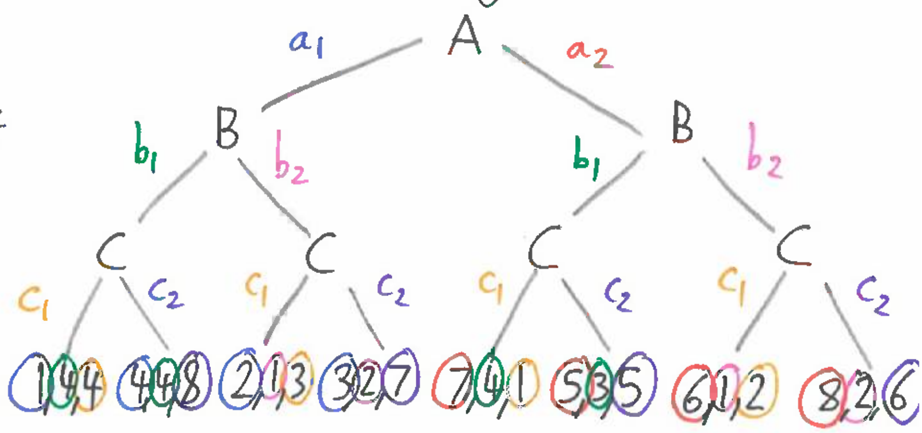
(d). You should choose Right (i.e. you should choose the right hand of player A) with complete certainty.
(This perhaps seems completely obvious, but it is interesting to consider whether you should maybe choose left with some possibility... it turns out that this would be sub-optimal and you should always choose right. We will see this idea in more detail in chapter 3).

2).

a). 3 players: A, B and C

b). Each player (A, B, C) has <u>two</u> pure strategies (a_1, a_2 for A, b_1, b_2 for B and c_1, c_2 for C).

c). Optimal play in this game is for player A to play strategy a_2, player B to play strategy b_1 and player C to play strategy c_2, awarding each player with payoffs of 5, 3 and 5 respectively.
Let's justify this.



Notice from the figure drawn, if player A plays a_2, then <u>regardless</u> of what player's B or C do, player A will receive one of the <u>red</u> circled payoffs. If player A plays a_1, they receive a <u>blue</u> circled payoff. But the lowest red payoff, 5, is <u>greater than</u> the highest blue payoff, 4. Thus there is no incentive for A to play a_1, so A plays a_2.

③

Similarly, if B plays b_1, they recieve a <u>green</u> circled payoff, the minimum of which is 3, but if B plays b_2 they recieve a pink payoff, with maximum 2. Since $3>2$, B plays b_1 is always optimal.

Lastly, a similar argument shows the purple payoffs are always greater than the orange payoffs for C, so C plays c_2.

3).

a). Let's denote the pure strategies by:

a_1, b_1 : Play 1 first. (Playing 2 second comes consequentially).

a_2, b_2 : Play 2 first.

a_3, b_3 : Copy the black card.

a_4, b_4 : Differ from the black card.

b).

B

A	b_1	b_2	b_3	b_4
a_1	0,0	0,0	$-\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$
a_2	0,0	0,0	$-\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$
a_3	$\frac{1}{2}, -\frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$	0,0	1,−1
a_4	$-\frac{1}{2}, \frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$	−1,1	0,0

The diagonal clearly gives a drawn game as the strategies are the same. Each other entry needs to be calculated, e.g. if A plays a_1, B plays b_2, then:

A wins

Prob. black card is 1 first

$$g_A(a_1, b_1) = 1 \times P(BC = 1 \text{ first}) + (-1) \times P(BC = 2 \text{ first}) = \frac{1}{2} - \frac{1}{2} = 0$$

A loses

Prob. black card is 2 first.

⋮

etc.

The game is <u>symmetric</u> (i.e. if we swap players A and B we have the same game, so we only need to calculate payoffs on one side of the diagonal. We can then reflect them.

c). They always sum to ~~$\frac{1}{2}$~~ 0. This is to be expected since in this ~~a~~ two-player game, either the game is drawn; both players get 0, or if one player wins, the other loses, so whoever gets +1 the other gets −1. In all cases the sum of the payoffs are zero.

(This is known as a zero-sum game. We usually simplify the normal form representation by omitting player B's entry in these games – since we know it is just $-g_A$. We'll study these games specifically in more detail in chapter 4. For now let's treat them as we will other games in normal form).

d). Notice that strategy a_3 strictly <u>dominates</u> a_1, since:

$$g_A(a_3,b_1) = \frac{1}{2} > 0 = g_A(a_1,b_1)$$
$$g_A(a_3,b_2) = \frac{1}{2} > 0 = g_A(a_1,b_2)$$
$$g_A(a_3,b_3) = 0 > -\frac{1}{2} = g_A(a_1,b_3)$$
$$g_A(a_3,b_4) = 1 > \frac{1}{2} = g_A(a_1,b_4),$$

So we can remove strategy a_1 for player A. By the symmetry of the game, we can remove strategy b_1 for player B.

A similar argument as above shows that a_3 <u>strictly dominates</u> a_2, and a_4 (so similarly b_3 dominates b_2 and b_4), so we can remove all of these strategies from the game.

This leaves only the pair of strategies (a_3,b_3) giving a 0 payoff to each player. This is optimal play in 2-card Goofspiel (i.e. both players copying the black card).

4).

a). A pure strategy for a player (let's say A) needs to tell them:

(1) which card to play in response to the first dealt black card;

and

(2) having played the first round, which second card to play in response to the second black card dealt and B's first played card.

(The third card to be played is then forced as it is the remaining card).

(1) is determined by a map:

$$\{1,2,3\} \longmapsto \{1,2,3\}$$

$$1^{st} \text{ black card} \longmapsto \text{A's first red card}$$

There are $\underline{\underline{3^3}}$ such maps.

(2) For each such map in (1), A now needs to know how to respond in the second round; i.e. for each i, j, where:

$i =$ B's first played card,

$j = 2^{nd}$ black card dealt,

player A has 2 remaining choices for the card to play. There are 3 possibilities for i and 2 for j, giving $3 \times 2 = 6$ situations. Thus, for each map in (1), there are then $\underline{\underline{2^6}}$ possible maps in (2).

$\Rightarrow$ In total, either player has $3^3 \times 2^6 = \underline{\underline{1728}}$ pure strategies available in 3-card Goofspiel!!

⑥

(b). Intuitively, maybe based on our findings for 2-card Goofspiel, we might guess that playing the pure strategy "copy the black card" constitutes optimal play in 3-card Goofspiel too.

Let's show that this is indeed the case.

Suppose player B is using this strategy, let's show that player A has no incentive to deviate from also using this strategy.

If player B is following the black card, then player B plays 3 when the black 3 is dealt. If player A loses this round, then player A <u>must</u> win the remaining two rounds to tie the game. This <u>is</u> possible, since A can play 3 against 2 (where B plays 2) and 2 against 1 (where B plays 1), assuming A plays 1 against 3.

Notice this strategy only gets a tie for the game, which is <u>not</u> an improvement on simply copying B's strategy and copying the black card.

Further, should A deviate to this strategy, then B can force a win against A by playing 2 against the 3 and 3 against the 1, rendering A's strategy 'sub-optimal'.

So, against 3 ~~our~~ we should play our 3 and copy the black card. Indeed it is then clear to see that we should play 2 against 2 and always copy the black card.

5). (◇)

- Player B can play the '1-up' strategy, i.e. against the black card x, play $x+1$ (where A will play x), and against n, play 1. This guarantees player B will win every round except for the round against n. Player B will have $\frac{(n-1)n}{2}$ many points, and player A will have n points.

Player B wins if $\frac{(n-1)n}{2} > n$

$$\Leftrightarrow n^2 - 3n > 0$$

$$\Leftrightarrow n(n-3) > 0$$

$$\Leftrightarrow n \geq 4 \text{ for } n \in \mathbb{Z}^+.$$

(Note that although this strategy defeats copying the black card in $n \geq 4$ card Goofspiel, it is defeated by the '2-up' strategy - you can gather what this would be)

- For $n \geq 4$, no pure strategy is optimal (any particular pure strategy can be defeated by another - in a similar way to what is discussed above), so mixed strategies (see chapter 3) are required.
The paper uploaded with these solutions has developed an algorithm that plays Goofspiel up to $n=13$ "optimally", the paper gives some discussion on what are possible optimal first moves based on the black card.
- The case $n=4$ is not really discussed in their paper and I think this is an interesting small case to examine in more detail.

6). (☆)

The second paper uploaded with these solutions gives a short proof that your optimal play is to <u>match the black card</u> against a random opponent. The original proof of this result is far longer and more technical; you can look this up if you're super interested.