Intro to Game Theory: Problem Set 2 Solutions

1). Circling in red the best response payoffs for player A and boxing in blue the best response payoffs for player B we find:

a).

A \ B	b_c	b_d
a_c	$-1,-1$	$-10,\boxed{0}$
a_d	$0,-10$	$-5,\boxed{-5}$

b).

A \ B	b_1	b_2
a_1	$2,\boxed{2}$	$0,1$
a_2	$1,0$	$1,\boxed{1}$

c).

A \ B	b_1	b_2
a_1	$-10,-10$	$2,\boxed{0}$
a_2	$0,\boxed{2}$	$0,0$

d).

A \ B	s_1	s_2
s_1	$0,\boxed{0}$	$2,\boxed{0}$
s_2	$0,\boxed{2}$	$1,1$

a). The prisoner's dilemma has only (a_d, b_d) (both player's defecting) giving payoffs of $-5,-5$ as a pure strategy equilibrium.

b). The trust dilemma has <u>two</u> pure strategy equilibria; both players trusting (a_1, b_1) receiving payoffs of $2,2$ or both players not-trusting (a_2, b_2) receiving payoffs of $1,1$.

c). The game of chicken also has <u>two</u> pure strategy equilibria. This time when the players have anti-synergy i.e. when one player swerves and the other drives straight (a_2, b_1) and (a_1, b_2) giving payoffs of ~~2~~ $0,2$ and $2,0$.

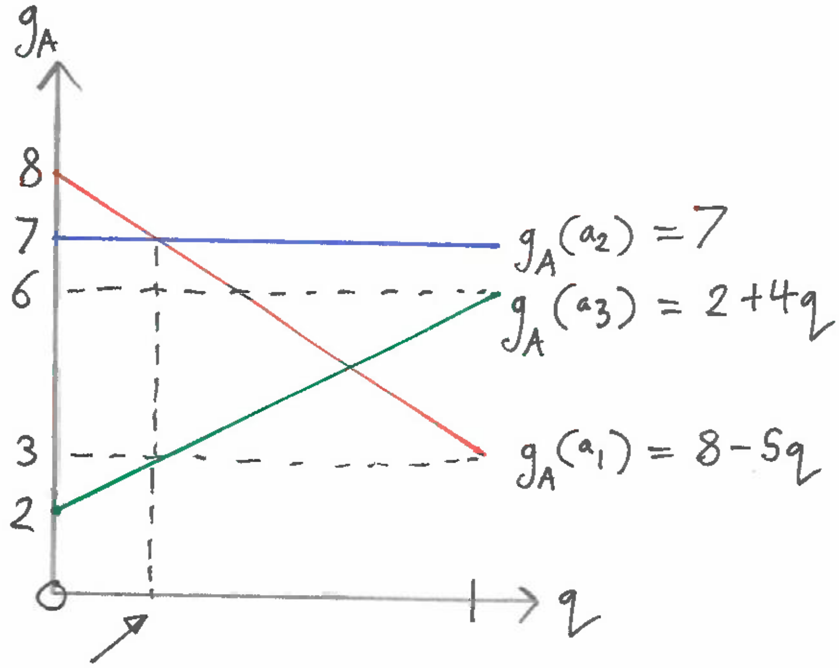
d). The sweets dilemma interestingly has <u>three</u> pure strategy equilibria given by (s_1, s_1), (s_1, s_2) and (s_2, s_1) with payoffs $0,0$, $2,0$ or $0,2$ respectively.

2). ②

a). Expected payoffs: $g_A(a_1) = 3q + 8(1-q) = 8 - 5q$

$g_A(a_2) = 7q + 7(1-q) = 7$

$g_A(a_3) = 6q + 2(1-q) = 2 + 4q$



$7 = 8 - 5q$

$\Rightarrow \underline{\underline{q = \frac{1}{5}}}$

b). • If $q < \frac{1}{5}$, then player A's best response is to shoot to the left. (play a_1).

• If $\frac{1}{5} < q$, then player A's best response is to shoot down the middle (play a_2).

• when $q = \frac{1}{5}$, then <u>either</u> of the above is a best response of player A.

• Player A should <u>never</u> shoot to the right (infact you can see that a_2 <u>dominates</u> a_3 here).

c). (◇) One idea that came to my mind was a tennis situation: you have just seen your opponent come forward to the net; you could try a passing shot to their left, their right or try a lob shot. Your opponent might check to their left, their right, or drop back centrally.

3). Consider an N-player game G in strategic form. Let S_i be the strategy set of player i and let s_i be a <u>dominated</u> strategy of player i and $\hat{s}_j$ be a <u>dominated</u> strategy of player j (where we allow for $i=j$ so these could be strategies of the same player).

Let G' be the game obtained from G by deleting s_i from S_i and $\hat{G}$ the game obtained from G by deleting $\hat{s}_j$ from S_j.

Suppose s_i is dominated by t_i and $\hat{s}_j$ is dominated by $\hat{t}_j$. Now if $\hat{t}_j \neq s_i$, then $\hat{s}_j$ is also dominated in G' by $\hat{t}_j$, and if $\hat{t}_j = s_i$ (meaning $i=j$) then $\hat{s}_j$ is also dominated in G' by t_i. Hence $\hat{s}_j$ is dominated in G'.

Similarly, if $t_i \neq \hat{s}_j$, then s_i is dominated in $\hat{G}$ by t_i, and if $t_i = \hat{s}_j$, then s_i is dominated in $\hat{G}$ by $\hat{t}_j$. So s_i is dominated in $\hat{G}$.

Removing these dominated strategies in G' and $\hat{G}$ results in the same game where both s_i and $\hat{s}_j$ have been removed from the game. Thus, the order doesn't matter. Similarly, removing both simultaneously ~~with~~ will also result in this same game. □

④

4).

a). a_2 is weakly dominated by either of a_1 or a_3 for A.

b_1 is weakly dominated by b_3 for player B.

b). a_2 weakly dominated by a_1 or a_3 $\rightarrow$ let's delete ~~a~~ a_2.

		B	
	b_1	b_2	b_3
A a_1	1,0	3,1	1,1
A a_3	2,2	3,3	0,2

b_3 weakly dominated by b_2 $\rightarrow$ let's delete b_3

	B	
	b_1	b_2
A a_1	1,0	3,1
A a_3	2,2	3,3

a_1 weakly dominated by a_3 $\rightarrow$ let's delete a_1

	B	
	b_1	b_2
A a_3	2,2	3,3

b_2 dominates b_1 $\rightarrow$ let's delete b_1

This gives us the pair of strategies (a_3, b_2) with payoffs 3,3 for the players c). b_1 weakly dominated by b_3 → let's delete b_1

		B	
		b_2	b_3
	a_1	3,1	1,1
A	a_2	3,0	0,1
	a_3	3,3	0,2

a_3 weakly dominated by a_1 → let's delete a_3

		B	
		b_2	b_3
A	a_1	3,1	1,1
	a_2	3,0	0,1

b_2 weakly dominated by b_3 → let's delete b_2

		B
		b_3
A	a_1	1,1
	a_2	0,1

a_2 dominated by a_1 → let's delete a_2

This gives us the pair of strategies (a_1, b_3) with payoffs 1,1 for the players.

d). The equilibria in the game are (a_3, b_2), (a_1, b_3) and (a_1, b_2). You can determine these in a similar manner to question 1 by circling and squaring the best response payoffs.

5).

a). Let's denote the strategy sets of both players by:

(i). $A_S = B_S = \{p_1, p_2, p_3, \cdots, p_{10}\}$, where p_j represents that the candidate chooses to stand at position j on the spectrum. Now observe that

$$g_A(p_1, p_1) = 50 < g_A(p_2, p_1) = 90$$
$$g_A(p_1, p_2) = 10 < g_A(p_2, p_2) = 50$$
$$g_A(p_1, p_3) = 15 < g_A(p_2, p_3) = 20$$
$$\vdots$$
$$g_A(p_1, p_{10}) = 50 < g_A(p_2, p_{10}) = 55,$$

which means that p_2 <u>strictly dominates</u> p_1 (for player A, but due to the symmetry the same would of course be true for player B).

Similarly, on the other side we could show p_9 <u>strictly dominates</u> p_{10}.

Thus we can <u>delete</u> strategies p_1 and p_{10} from both players strategy sets, so now $A_S' = B_S' = \{p_2, p_3, \cdots, p_9\}$.

Now that p_1 and p_{10} have been deleted, p_3 <u>strictly dominates</u> p_2 (and p_8 strictly dominates p_9), so we can delete these. Continuing this process of iteratively deleting dominated strategies we arrive at the game where $\hat{A}_S = \hat{B}_S = \{p_5, p_6\}$ with normal form:

		B	
		p_5	p_6
A	p_5	50, 50	50, 50
	p_6	50, 50	50, 50

clearly all four pairs of pure strategies are pure equilibria in this game, and so we conclude that (p_5, p_5), (p_5, p_6), (p_6, p_5) and (p_6, p_6) are all the pure strategy equilibria in the two-player election game. They all give 50% of the votes to each player.

(ii). $(\Diamond)$

b). $(\Diamond)$

(__Remark:__ Albeit a very simplified model of an election, the phenomenon of candidates 'grouping' towards the centre in politics is a __very common__ occurrence, known as the __median voter theorem__. Many historical famous elections have been won by candidates taking their party (or at least making it look!) towards the centre. The last 'strong' Labour government in the UK was elected when Tony Blair took the party towards the centre ... more recently Jeremy Corbin arguably took the party further left than usual which alienated large portions of the voters resulting in the government we have now being elected ... which I make no personal opinion on here but encourage you to discuss (I'll teach you the game theory, you can debate the politics!))

6). First observe that when __nobody__ contributes we have an equilibrium of the game: no single player can deviate to enable the fountain to be built because $K \geqslant 2$, so therefore 'not contributing' is a best response.

Secondly we claim that any other equilibrium is of the form where <u>exactly K people contribute</u>. This <u>is</u> an equilibrium since a player who does not contribute currently gains 2 sweets, but would only gain 1 sweet upon contributing and a player who is contributing and currently gaining 1 sweet would gain nothing if they decided to not contribute. So everyone is playing their best responses.

Any other strategy profile is <u>not</u> in equilibrium: Suppose more than K people contribute, then each of them (individually) would gain an additional sweet by not contributing, because upon their unilateral deviation the fountain would still be built. Suppose fewer than K people contribute, so the fountain is not built, then each person who contributed would do better (lose nothing rather than lose a sweet) by not contributing.

7).

a). Payoff to A: $g_A(x,y) = x(12-x-y) - x = x(11-x-y)$

Payoff to B: $g_B(x,y) = y(12-x-y) - 2y = y(10-x-y)$

By completing the square or calculus, the best response of firm A to strategy y from firm B is $x = \frac{11}{2} - \frac{y}{2}$, which will be non-negative since $y < 11$ otherwise firm B's payoff becomes negative.

Similarly, the best response of firm B to x is to play $y = 5 - \frac{x}{2}$, which will also be non-negative since we know $x \leq \frac{11}{2}$ by the previous calculation.

An equilibrium is then a pair of <u>mutual</u> best responses, found via solving

$$x = \frac{11}{2} - \frac{y}{2} \quad \text{and} \quad y = 5 - \frac{x}{2} \quad \text{simultaneously, giving}$$

$$x = \frac{11}{2} - \frac{1}{2}\left(5 - \frac{x}{2}\right), \quad \text{or} \quad x = 4 \text{ and therefore}$$

$$y = 3, \text{ defining equilibrium}$$

The payoffs in equilibrium are

$$g_A(4,3) = 4(11-4-3) = 16 \text{ to firm A, and}$$

$$g_B(4,3) = 3(10-4-3) = 9 \text{ to firm B.}$$

b). (◇)

The game with integer quantities for x and y has <u>three</u> equilibria at $(4,3)$ (as found in (a)), but also at $(3,4)$ and at $(5,2)$.

c). (◇)

8).

a).

		B	
		s_1	s_2
A	s_1	u,u	x,y
	s_2	y,x	v,v

, where u, v, x, y could be distinct is the most general this game could be. 4 payoffs (rather than ~~[illegible]~~ 8) fully specify the game.

b). (☆)

(i). Due to the symmetry of the game any strategy profile (set of strategies; one for each player) is simply determined by how many players choose S_1, say k players $(0 \leq k \leq N)$, and then the rest choose S_2. Due to the symmetry of the game, which individual players choose S_1 doesn't matter, only the total amount of players that choose S_1, so we need, for each value of $k = 1, 2, \ldots, N-1$, just two payoffs: one for each player playing S_1 and another for each player playing S_2. This is $2(N-1)$ payoffs, but we also need an extra payoff when everybody plays S_1 and one when everybody plays S_2, giving us $2N$ needed payoffs.

(ii). First suppose S_1 dominates S_2. Then $(S_1, S_1, \ldots, S_1)$ is clearly the unique equilibrium of the game, so suppose this is not the case.

Then, because S_2 is not dominated by S_1 and there are only two strategies, for some profile of strategies of the other players, the payoff to a given player when playing S_2 is greater than or equal to the payoff when playing S_1, i.e. S_2 is a best response to the remaining strategies of the other players.

Consider a profile of strategies given by (one for each player):

$$(\underbrace{S_1, S_1, \ldots, S_1}_{k}, \underbrace{S_2, \ldots, S_2}_{N-k}),$$

where S_2 is a best response, where k above is the smallest number possible $(0 \leq k < N)$ such that S_2 is a best response. Then this profile is in equilibrium: indeed, by assumption, S_2 is a best response to the other player's strategies for every player playing S_2.

But for the player's playing s_1, s_1 is currently a best response to the strategy profile, since, if s_1 were not a best response for these players, then they could do better by changing their strategy to s_2, but this contradicts our assumption that we have chosen the smallest possible k where s_2 is a best response. □

Hence the game has an equilibrium.

(iii). Part (ii) has already proved this!

($\Rightarrow$): If s_1 dominates s_2, there is no incentive to play s_2 for any player, so $(s_1, \dots, s_1)$ is a unique equilibrium.

($\Leftarrow$): Consider the contrapositive of the statement:

$(s_1, \dots, s_1)$ unique equilibrium $\Rightarrow$ s_1 dominates s_2

which is:

s_1 does not dominate s_2 $\Rightarrow$ $(s_1, \dots, s_1)$ is not the unique equilibrium.

In part (ii) we proved that if s_1 did not dominate s_2, then we had equilibria of the form $(\underbrace{s_1, \dots, s_1}_{k}, \underbrace{s_2, \dots, s_2}_{N-k})$, where $k \neq N$.

So we are done. □

c). • ✓ all four games specified by four payoffs.

• ✓ all four games had a pure strategy equilibrium.

• ✓ The only game which had one strategy dominate the other was the Prisoner's dilemma, and indeed this was the only game with a unique equilibrium with both players playing the dominating strategy.

d). Matching pennies is <u>not</u> symmetric, interchanging the players and their strategies does not result in the same game (A is trying to match, B not-match).

Rock-paper-scissors <u>is</u> a symmetric game, but each player has <u>three</u> pure strategies not two, hence why the results break down.