MATH60141/70141 Introduction to Game Theory — Dr Sam Brzezicki

Problem Set 2

1). Determine all pure strategy equilibria in the following two-player games:

a). **The Prisoner's Dilemma:** See section 2.1 of the lecture notes.

b). **The Trust Dilemma:** Two players can each guarantee getting a payoff of 1, but have the opportunity to get 2 if they can trust each other (this game is often called 'The Stag Hunt' - referring to two hunters who can work together to catch a prized stag, or hunt a simple hare themselves).

c). **Chicken:** Two racers are driving toward each other at high speed. They can choose to continue to drive straight, or to 'chicken out' and swerve away. Both want to look cool by not flinching, but the consequences are great if they crash into one another.

d). **The Sweets Dilemma:** Played in class; see section 1.3 of the lecture notes.

A \ B	b_1	b_2
a_1	2, 2	0, 1
a_2	1, 0	1, 1

Figure 1: (b). The Trust Dilemma.

A \ B	b_1	b_2
a_1	-10, -10	2, 0
a_2	0, 2	0, 0

Figure 2: (c). Chicken.

2). Consider the variant of the penalty kick game shown below where the attacker, player A, has a particularly powerful but less accurate shot. Their payoffs are improved when shooting down the middle, but drop when required to position the ball to the sides. Our player is right-footed, meaning that they have slightly better accuracy when pulling the ball to the left of the goal rather than aiming to the right of the goal. As in class, the payoffs for A are ten times the probability that they score a goal and the goalkeeper's payoffs are negative A's payoffs.

a). Sketch a graph showing the expected payoffs for each of player A's pure strategies against q; the probability that the goalkeeper dives to the left.

b). Describe player A's best response for the different values of q.

c). ($\diamond$) For the sports player's among you; perhaps you can model a similar competitive scenario in your sport.

A \ B	Left b_1	Right b_2
Left a_1	3,-3	8,-8
Middle a_2	7,-7	7,-7
Right a_3	6,-6	2,-2

Figure 3: A Penalty kick game for Q2.

3). Prove that when deleting strictly dominated strategies from a game the order in which they are removed does not matter and that several dominated strategies can be removed simultaneously.

4). Consider the two-player 3×3 game shown below.

A \ B	b_1	b_2	b_3
a_1	1,0	3,1	1,1
a_2	1,1	3,0	0,1
a_3	2,2	3,3	0,2

a). Identify all weakly dominated strategies in the game.

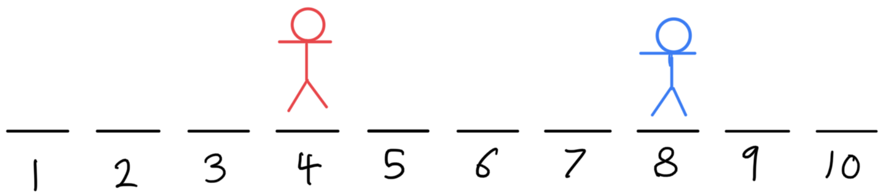
b). Let's now assume that we **can** remove weakly dominated strategies of either player from the game. Do this, and repeat this process (of iterated deletion of weakly dominated strategies) until you find a single strategy pair of the original game (you may find you need to swap the order in which you delete strategies to arrive at a single strategy pair).

c). Now perform a different order of iterated deletion of weakly dominated strategies that results in another single strategy pair which is different from the one you found in (b).

d). Find all pure strategy equilibria in the game.

(**Remark:** This exercise shows that deleting weakly dominated strategies, and iterating this, is not a justifiable process)

5). **The Election game:** Consider the political spectrum modelled by a collection of discrete positions in a line, numbered 1 through 10, as shown in the figure below.



Position 1 represents the far-left wing stance and position 10 the far right-wing stance. In this game our players are political candidates trying to maximise getting as much of the public vote as possible. The players will all, simultaneously, choose a position in which to stand on the spectrum (multiple players at the same position is allowed).

We will suppose that 10% of the voters in this election lie at each position on our spectrum and that voters will vote for the closest candidate to their position. In the case of a tie between closest candidates, the votes are split proportionally between the tied candidates (for example, with one candidate at position 1, none at position 2 and two candidates at position 3, then the votes for the 10% of voters at position 2 would be split as 5% to the candidate at position 1 and 2.5% to each of the two candidates at position 3).

We will take the payoffs of our candidates to be their % share of all the votes, so, for example, in the figure above with 2 candidates in the game, the red candidate would receive all the votes from positions 1 through 5, as well as half the votes from position 6, giving the a payoff of 55. The blue candidate would get a payoff of 45.

a). Determine all pure strategy equilibria in the game when there are:

(i). 2 players;

(ii). ($\diamond$) N players for different values of $N \geqslant 3$.

b). ($\diamond$) How would the game change if we considered a continuous spectrum instead? What about a normal distribution of voters across the spectrum?

6). **The Sweet Fountain:** Each of N players simultaneously chooses whether or not to contribute 1 sweet (and no more) towards the building of a sweet fountain.

The sweet fountain will be constructed if and only if at least K players contribute (so it requires at least K sweets to build), where K is a constant such that $2 \leqslant K \leqslant N$.

If the fountain is not built, the contributions made will not be refunded to those players that made them. If the fountain is built then it generates 2 sweets for each player in the game, regardless of whether or not they contributed to building it (so players that contributed will net a 1 sweet gain, those that didn't contribute get a 2 sweet gain).

Modelling the payoffs for the players in the game by the net number of sweets gained (which could be negative), find, with justification, all pure strategy equilibria in this game.

7). Consider the **Cournot duopoly** game as studied in section 2.9 where firms A and B produce quantities x and y ($x, y \in [0, 12]$) of a product where the unit price on the market is $12 - x - y$.

Suppose now that the firms incur a cost of production, which is a cost of 1 per unit produced for firm A and a cost of 2 per unit produced for firm B.

a). Find the payoffs to the firms in this game and determine the equilibrium of this game. What are each firm's profits in equilibrium?

b). ($\diamond$) Suppose now that x, y must take only integer values (they represent the exact numbers of the individual products to be produced which are indivisible). What are the equilibria in the game in this case?

c). ($\diamond$) Investigate the Bertrand model of competition (named after the French mathematician Joseph Bertrand).

8). We start with a definition. **Definition:** An N-player game is **symmetric** if each player has the same set of strategies and if the game remains the same after any permutation of the players and their payoffs.

a). In a two-player symmetric game where both players have two pure strategies, give a normal form representation of the game in as much generality as possible. How many payoffs fully specify the game?

b). ($\star$) Consider now an N-player symmetric game where each player has two pure strategies (let's call them s_1 and s_2).

(i). Explain why the game is fully specified by just $2N$ payoffs.

(ii). Prove that this game has a pure strategy equilibrium.

(iii). Prove that the strategy profile $(s_1, s_1, \cdots, s_1)$ (i.e. all players playing s_1), is the unique equilibrium of the game if and only if s_1 dominates s_2.

c). Return to the four games in question 1 and verify that the games satisfy the results from part (b) with $N = 2$.

d). In class we saw that the games of **matching pennies** and **rock-paper-scissors** (section 2.10) had no pure strategy equilibria. These however are two-player games with the same set of pure strategies for each player, so why do these games not invalidate (b)(ii) above?