1

Intro to Game Theory: Problem Set 3 Solutions

1).

a).

		B	
		b_1	b_2
A	a_1	2, 2	8, 1
	a_2	5, 3	1, 7

- No pure strategy equilibria. So we seek mixed strategy equilibria. Let $\alpha = (p, 1-p)$, $\beta = (q, 1-q)$ be mixed strategies for the players.
- Then:

$$g_A(a_1, \beta) = 2q + 8(1-q) = 8 - 6q$$

$$g_A(a_2, \beta) = 5q + (1-q) = 1 + 4q$$

Player A can be made indifferent $\Leftrightarrow$ $8 - 6q = 1 + 4q$ $\Leftrightarrow$ $q = \frac{7}{10}$.

So $\beta = \left(\frac{7}{10}, \frac{3}{10}\right)$ is an equaliser strategy for B. Similarly:

$$g_B(\alpha, b_1) = 2p + 3(1-p) = 3 - p$$

$$g_B(\alpha, b_2) = p + 7(1-p) = 7 - 6p$$

Player B can be made indifferent $\Leftrightarrow$ $3 - p = 7 - 6p$ $\Leftrightarrow$ $p = \frac{4}{5}$.

So $\alpha = \left(\frac{4}{5}, \frac{1}{5}\right)$ is an equaliser strategy for A.

Therefore (α, β) is a mixed equilibrium.

b).

		B		
		b_1	b_2	b_3
A	a_1	−1, −1	2, −3	1, −2
	a_2	1, 0	2, 0	−2, 1

- First observe that b_3 strictly dominates b_2, so we delete strategy b_2 from the game.
- Now checking the remaining 2×2 game we find no pure strategy equilibria, so, letting $\alpha = (p, 1-p)$, $\beta = (q, 1-q)$ we seek a mixed strategy equilibrium. We have:

②

$$g_A(a_1,\beta) = -q + 1 - q = 1-2q$$
$$g_A(a_2,\beta) = q - 2(1-q) = 3q-2$$

Player A can be made indifferent $\iff 1-2q = 3q-2 \iff \underline{q = \frac{3}{5}}$

So $\underline{\beta = \left(\frac{3}{5}, \frac{2}{5}\right)}$ is an equaliser strategy for B. Similarly:

$$g_B(\alpha, b_1) = -p \quad , \quad g_B(\alpha, b_2) = -2p + 1 - p = 1-3p .$$

Player B can be made indifferent $\iff -p = 1-3p \iff \underline{p = \frac{1}{2}}$

So $\underline{\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)}$ is an ES for A. Therefore $\underline{\left(\left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{3}{5}, 0, \frac{2}{5}\right)\right)}$ is an equilibrium in the original game.

c). I'll do this one using the method shown in section 3.6; checking subgames!

		B: b_1	b_2
	a_1	$0, \boxed{1}$	$6, 0$
A	a_2	$2, 0$	$5, \boxed{2}$
	a_3	$3, 3$	$3, \boxed{4}$

- No pure strategy equilibria. Let $\beta = (q, 1-q)$ be a mixed strategy of player B.
- First suppose player A doesn't play a_3 with any probability; so then $\alpha = (p, 1-p, 0)$ is a mixed strategy of player A. In this case an ES for A is when $p = \frac{2}{3}$. Similarly, an ES for B occurs when $q = \frac{1}{3}$. The expected payoff for player A is then 4 for both a_1 and a_2, which is <u>higher</u> than the expected payoff of 3 they would get for playing a_3. So indeed; $\underline{\left(\left(\frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)}$ is an equilibrium.

- Next we seek a mixed strategy of the form $\alpha = (p, 0, 1-p)$ for player A. This time an ES for A occurs when $p = \frac{1}{2}$. An ES for B occurs when $q = \frac{1}{2}$. The expected payoff for A is then 3 for each of a_1 and a_3, but note that when A plays a_2 against this β they get expected payoff of $\frac{7}{2} > 3$, so <u>no</u> equilibria are found in this subgame.

- Finally, consider the case where player A plays $\alpha = (0, p, 1-p)$. This time $g_B(\alpha, b_1) = 3 - 3p$ and $g_B(\alpha, b_2) = 2p + 4(1-p) = 4 - 2p$. Upon attempting to equate these payoffs we arrive at an impossibility (this is because with a_1 removed then b_2 dominates b_1), so clearly there are no equilibria in this subgame.
- There is no strategy where player A can mix over all 3 pure strategies, hence the game has only <u>one</u> equilibrium at $\left(\left(\frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$.

d).

		B		
		b_1	b_2	b_3
	a_1	1, 0	2, $\boxed{2}$	4, −1
A	a_2	−1, 1	0, −5	5, $\boxed{8}$
	a_3	−4, −2	−1, −1	8, $\boxed{3}$

- There are two pure strategy equilibria given by (a_1, b_2) and (a_3, b_3).
- Consider now, for instance, the mixed strategy $\alpha = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$ for player A. Well, we find:

$$g_A(\alpha, b_1) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3} > -1 = g_A(a_2, b_1)$$

$$g_A(\alpha, b_2) = \frac{4}{3} - \frac{1}{3} = 1 > 0 = g_A(a_2, b_2)$$

$$g_A(\alpha, b_3) = \frac{8}{3} + \frac{8}{3} = \frac{16}{3} > 5 = g_A(a_2, b_3),$$

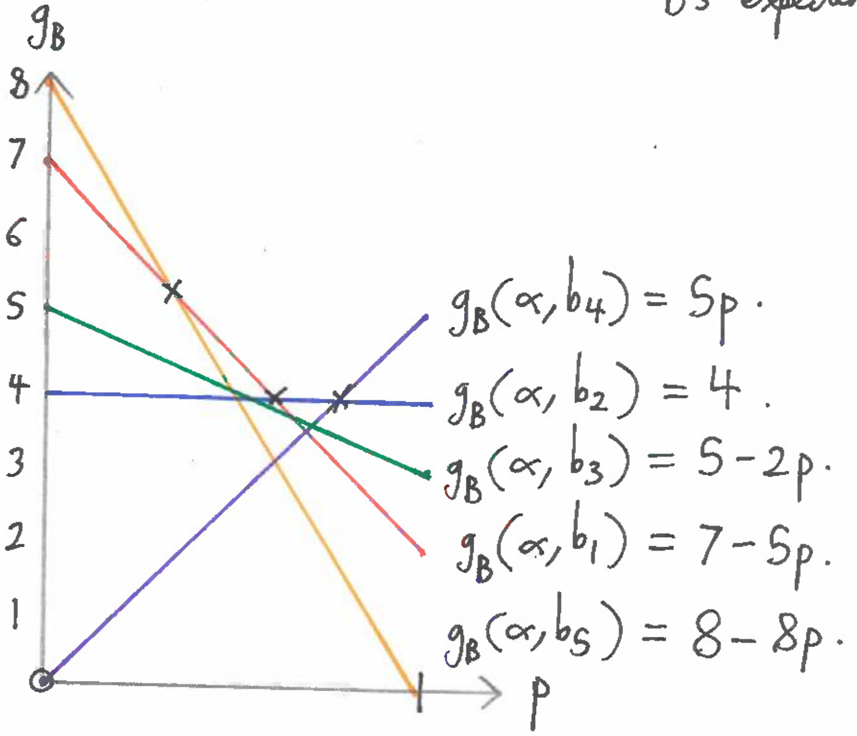
so <u>α strictly dominates a_2</u>, meaning we can <u>delete</u> a_2 from the game. With a_2 deleted, <u>b_2 strictly dominates b_1</u> for player B, so we can delete b_1 from the game. We arrive with the 2×2 game involving strategies $\{a_1, a_3\}$ for A and $\{b_2, b_3\}$ for B. Seeking a mixed equilibrium in this game one finds that $\alpha^* = \frac{4}{7} a_1 + \frac{3}{7} a_3$ and $\beta^* = \frac{4}{7} b_2 + \frac{3}{7} b_3$ form an additional equilibrium of the game.

4

e).

		B				
		b_1	b_2	b_3	b_4	b_5
A	a_1	0,2	2,4	1,3	0,$\boxed{5}$	3,0
	a_2	1,7	0,4	4,5	1,0	0,$\boxed{8}$

- The game has no pure strategy equilibria.
- Let $\alpha = (p, 1-p)$ be a mixed strategy for player A. We draw the upper-envelope diagram for player B's expected payoff against p.



We see the only cases where player B has two best responses occur at the 3 marked intersection points along the upper-envelope. We consider each in turn:

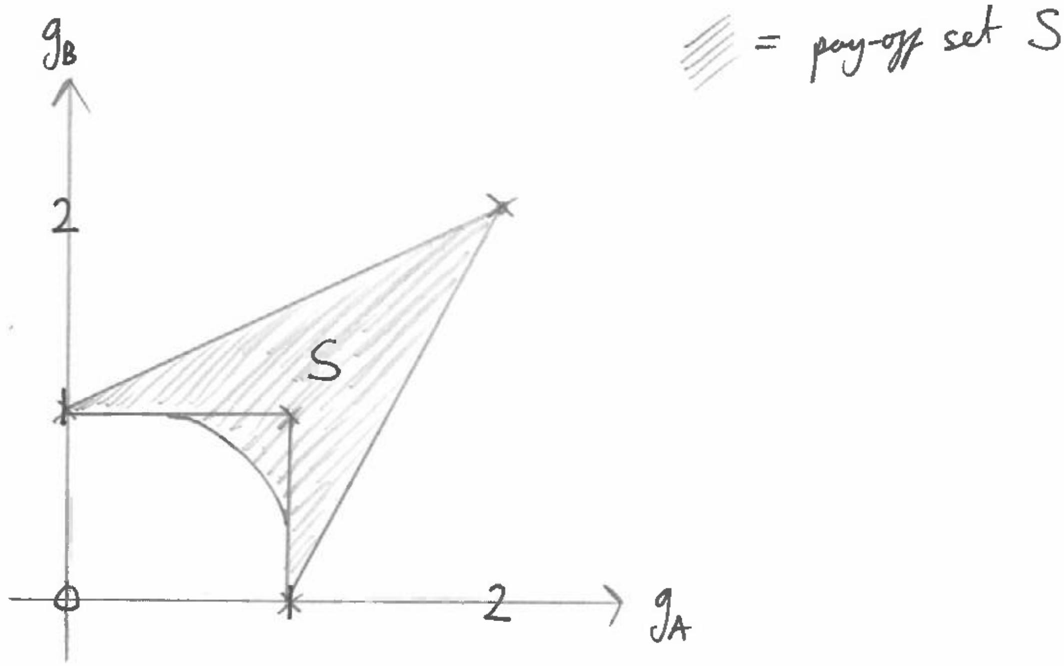
- when B plays only b_1 or b_5 then A can make B indifferent by choosing $p = \frac{1}{3}$, or playing $\alpha_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$. Against this player B can make A indifferent by playing $\beta_1 = \left(\frac{3}{4}, 0, 0, 0, \frac{1}{4}\right)$. (α_1, β_1) is the first equilibrium of the game.
- when B plays only b_1 or b_2 then A can make B indifferent by choosing $p = \frac{3}{5}$, or playing $\alpha_2 = \left(\frac{3}{5}, \frac{2}{5}\right)$. Against this player B can make A indifferent by playing $\beta_2 = \left(\frac{2}{3}, \frac{1}{3}, 0, 0, 0\right)$. (α_2, β_2) is a second equilibrium of the game.

⑤

- Finally when B plays b_2 or b_4 then A can make B indifferent by choosing $p = \frac{4}{5}$, or playing $\alpha_3 = \left(\frac{4}{5}, \frac{1}{5}\right)$. Against this player B can make A indifferent by playing $\beta_3 = \left(0, \frac{1}{3}, 0, \frac{2}{3}, 0\right)$. (α_3, β_3) is the third and final equilibrium of the game, since as the upper-envelope diagram shows, there are no other possibilities where player B can play more than one pure strategy as a best response.

2).

a).



b). Let $\alpha = (p, 1-p)$, $\beta = (q, 1-q)$ be mixed strategies of the players.

$g_A(a_1, \beta) = 2q$, $g_A(a_2, \beta) = 1$, so player A is indifferent when $q = \frac{1}{2}$. By the symmetry of the game B will be indifferent when $p = \frac{1}{2}$, so we conclude $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is a mixed equilibrium in the game.

3).

a).

		B	
		H	D
A	H	$\frac{V-C}{2}, \frac{V-C}{2}$	$V, \boxed{0}$
	D	$0, \boxed{V}$	$\frac{V}{2}, \frac{V}{2}$

• There are two pure strategy equilibria at (H,D) and (D,H). Seeking any mixed equilibria we let $\alpha = (p, 1-p)$, $\beta = (q, 1-q)$. Then:

$$g_A(H, \beta) = \left(\frac{V-C}{2}\right) q + V(1-q)$$
$$= V - \frac{1}{2}(V+C) q$$
$$g_A(D, \beta) = \frac{V}{2}(1-q) = \frac{V}{2} - \frac{V}{2} q$$

Thus, player A can be made indifferent when $q = \frac{V}{C}$. Since $0 < V < C$ this is a valid probability. By the symmetry of the game player B can be made indifferent when $p = \frac{V}{C}$. Thus $\left(\left(\frac{V}{C}, 1-\frac{V}{C}\right), \left(\frac{V}{C}, 1-\frac{V}{C}\right)\right)$ is a mixed equilibrium of the game. So there are 3 equilibria in the game.

b). Mixed equilibria in this game can often be thought of as stable population proportions ~~but~~ of the two species: A population of all Doves or all Hawks performs worse than when one member defects to the other type (think of how well an animal might perform in the game as affecting their birth rate of more of that species). At the mixed equilibrium however there is no incentive for either animal to 'want' to be ~~the~~ any other 'type' specifically, so this can be thought of as a stable population distribution ($100\frac{V}{C}$ % Hawks and $100\left(1-\frac{V}{C}\right)$% Doves).

[This concept might interest you and you are welcome to research more on this game and how we can model competition between species with it as coursework ideas!]

c). If $c = V$ then we get a Sweets dilemma, if $c < V$ we get a prisoner's dilemma.

d). (◇)

4).

a).

		B	
		b_1	b_2
A	a_1	$0, \boxed{0}$	$-10, \boxed{0}$
	a_2	$-1, \boxed{0}$	$-6, -90$

• There is one pure strategy equilibrium in the game at (a_1, b_1).

• Next observe that the game is degenerate because the pure strategy a_1 has two pure best responses, b_1 and b_2. Because player B is indifferent between b_1 and b_2 when A plays a_1, player B can mix between b_1 and b_2. This defines an equilibrium as long as a_1 stays a best response to B's mixing. a_1 is a best response to $\beta = (q, 1-q)$ if and only if: $-10(1-q) \geq -q - 6(1-q)$ i.e. $g_A(a_1, \beta) \geq g_A(a_2, \beta)$.

That is: $q \geq \frac{4}{5}$; therefore: $(a_1, (q, 1-q))$, where $\frac{4}{5} \leq q \leq 1$ are an infinite set of equilibria.

Suppose further there was a mixed strategy equilibrium where player A plays a_2 with some positive probability; i.e. $\alpha = (p, 1-p)$. Player B's best response to this α is always b_1 however, but then player A's best response to b_1 is a_1, so a_2 cannot have positive probability in any equilibrium.

⑧

b).

		b_1	b_2
A	a_1	$1, \boxed{1}$	$1, \boxed{1}$
	a_2	$0, \boxed{2}$	$2, 0$

• (a_1, b_1) is a pure strategy equilibrium.

• The game is degenerate: a_1 has two pure strategy best responses: b_1 and b_2.

• Let $\beta = (q, 1-q)$. Then a_1 is a best response to β if and only if:

$$g_A(a_1, \beta) \geq g_A(a_2, \beta) \iff 1 \geq 2(1-q) \iff q \geq \frac{1}{2}.$$

Thus: $(a_1, (q, 1-q))$, where $\frac{1}{2} \leq q \leq 1$ are an infinite set of equilibria.

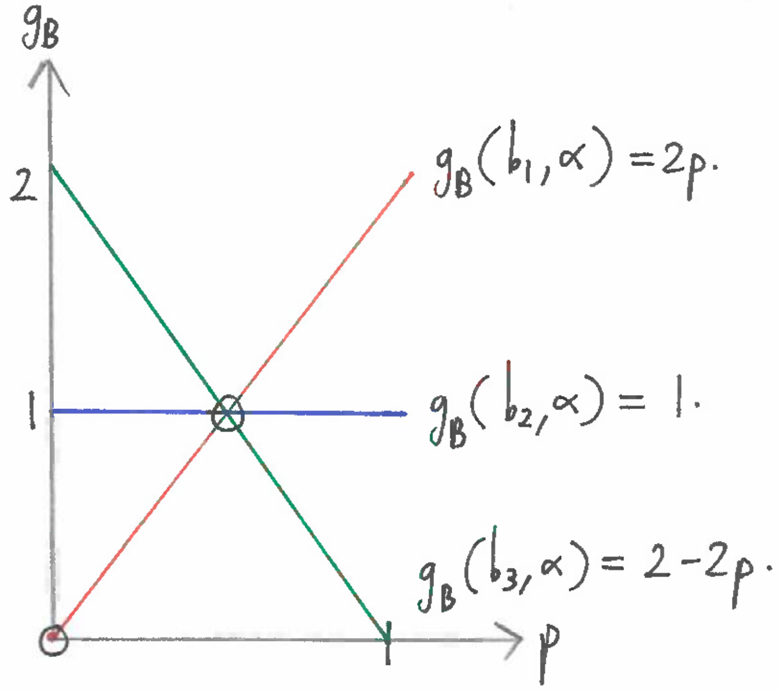
• No other equilibria as if a_2 played with any positive probability then B plays b_1, but then A should play a_1.

c).

		b_1	b_2	b_3
A	a_1	$0, \boxed{2}$	$3, 1$	$2, 0$
	a_2	$1, 0$	$2, 1$	$0, \boxed{2}$

• No pure strategy equilibrium.

• We draw the upper-envelope diagram for player B against p, where $\alpha = (p, 1-p)$ is a mixed strategy for player A.



• All three lines cross at a single point where all three pure strategies of player B are best responses when player A plays this mixed strategy $\alpha = (\frac{1}{2}, \frac{1}{2})$.

An equilibrium is only possible when both players mix, so this requires player A to play the mixed strategy $\alpha = (\frac{1}{2}, \frac{1}{2})$, otherwise player B has a unique pure strategy best response against A's mixed strategy but then A's mixed strategy is not a best response to this. So if A is playing $\alpha = (\frac{1}{2}, \frac{1}{2})$, then A needs to be made indifferent over each of their pure strategies. To achieve this player B can mix over <u>all three</u> of their best responses b_1, b_2 and b_3.

Let $\beta = (q_1, q_2, 1-q_1-q_2)$ be a mixed strategy for player B. Then:

$$g_A(a_1, \beta) = 3q_2 + 2(1-q_1-q_2) = 2 - 2q_1 + q_2$$
$$g_A(a_2, \beta) = q_1 + 2q_2$$

For indifference: $q_1 + 2q_2 = 2 - 2q_1 + q_2 \iff \underline{\underline{3q_1 + q_2 = 2}}$

So from this let's determine the extreme values of, say, q_2. Either $q_2 = 0$, so $q_1 = \frac{2}{3}$ or $q_2 = \frac{1}{2}$, so $q_1 = \frac{1}{2}$. Notice q_2 cannot be larger than $\frac{1}{2}$ otherwise ~~q_2 becomes~~ $q_1 + q_2 > 1$. Thus letting $q_2 = k \in [0, \frac{1}{2}]$ be a free parameter, we can conclude that all equilibria in the game are of the form:

$$\underline{\underline{\left(\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{2-k}{3}, k, \tfrac{1-2k}{3}\right) \right), \quad k \in [0, \tfrac{1}{2}]}}.$$

5). Let (α, β) be a mixed strategy equilibrium of the given nondegenerate game. Define the sets:

$$\mathrm{Supp}(\alpha) = \{ i : p_i > 0 \}, \quad \mathrm{Supp}(\beta) = \{ j : q_j > 0 \},$$

known as the <u>supports</u> of the mixed strategies $\alpha = (p_1, p_2, \dots, p_n)$, $\beta = (q_1, q_2, \dots, q_m)$.

Let $BR(\alpha) = \{ j : b_j$ is a pure best response to $\alpha \}$,

$BR(\beta) = \{ i : a_i$ is a pure best response to $\beta \}$.

The best response condition can then be written as the set inclusions:

$$\mathrm{Supp}(\alpha) \subseteq BR(\beta) \quad \text{and} \quad \mathrm{Supp}(\beta) \subseteq BR(\alpha).$$

Further, the nondegeneracy condition can be stated as

$$|BR(\alpha)| \leqslant |\mathrm{Supp}(\alpha)| \quad \text{and} \quad |BR(\beta)| \leqslant |\mathrm{Supp}(\beta)|.$$

This implies that:

$$|BR(\alpha)| \leqslant |\mathrm{Supp}(\alpha)| \leqslant |BR(\beta)| \leqslant |\mathrm{Supp}(\beta)| \leqslant |BR(\alpha)|,$$

and so all inequalities hold as equalities and $|BR(\alpha)| = |BR(\beta)|$ as we needed to prove. $\square$

Note that this also proves that every pure best response is played with positive probability in the equilibrium: there are no unused best responses.

6). Let $\alpha^* = k\alpha + (1-k)\hat{\alpha}$. Clearly, $\alpha^* \in A_S$ because, denoting $\alpha^* = (p_1^*, p_2^*, \dots, p_n^*)$, then $p_i^* \geqslant 0$ for all i, and:

$$\sum_{i=1}^{n} p_i^* = \sum_{i=1}^{n} \left(k p_i + (1-k)\hat{p}_i \right) = k \sum_{i=1}^{n} p_i + (1-k) \sum_{i=1}^{n} \hat{p}_i$$

$$= k + (1-k)$$

$$= 1,$$

where we let $\alpha = (p_1, p_2, \dots, p_n)$, $\hat{\alpha} = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n)$.

The pair (α^*,β) is an equilibrium if α^* is a best response to β and vice-versa. For any mixed strategy $\tilde{\alpha}$ of player A, we have:

$$g_A(\alpha,\beta) \geqslant g_A(\tilde{\alpha},\beta) \quad \text{and} \quad g_A(\hat{\alpha},\beta) \geqslant g_A(\tilde{\alpha},\beta), \quad \text{consequently:}$$

$$\begin{aligned} g_A(\alpha^*,\beta) &= k\,g_A(\alpha,\beta) + (1-k)\,g_A(\hat{\alpha},\beta) \\ &\geqslant k\,g_A(\tilde{\alpha},\beta) + (1-k)\,g_A(\tilde{\alpha},\beta) = g_A(\tilde{\alpha},\beta), \end{aligned}$$

which shows that α^* is a best response to β. Similarly, β is a best response to α and $\hat{\alpha}$ (can be proved similarly to the above), hence (α^*,β) is an equilibrium, as claimed. $\square$

7). $(\star)(\diamond)$