Problem Set 3

1). Determine **all** equilibria (meaning both pure and mixed) in the following two-player games:

a).

A \ B	b_1	b_2
a_1	2,2	8,1
a_2	5,3	1,7

b).

A \ B	b_1	b_2	b_3
a_1	-1,-1	2,-3	1,-2
a_2	1,0	2,0	-2,1

c).

A \ B	b_1	b_2
a_1	0,1	6,0
a_2	2,0	5,2
a_3	3,3	3,4

d).

A \ B	b_1	b_2	b_3
a_1	1,0	2,2	4,-1
a_2	-1,1	0,-5	5,8
a_3	-4,-2	-1,-1	8,3

e).

A \ B	b_1	b_2	b_3	b_4	b_5
a_1	0,2	2,4	1,3	0,5	3,0
a_2	1,7	0,4	4,5	1,0	0,8

2). Consider the **trust dilemma** as seen on problem set 2 with normal form as shown in the figure below.

		B	
		b_1	b_2
A	a_1	2, 2	0, 1
	a_2	1, 0	1, 1

Figure 1: The Trust Dilemma.

a). Give a sketch of the payoff set for the game.

b). Find all mixed equilibria in the game (Problem Set 2, Q1(b) tasked you with finding the pure equilibria already).

3). **Hawks and Doves:** Let's consider a game of conflict between two different animal species: hawks, H, and doves, D. The species are in conflict over a contested resource, which might be food, nesting grounds, etc, which is worth a value $v > 0$ to the animals.

When two doves come into conflict over the resource they peacefully split the resource equally. When a hawk meets a dove, the dove stands off and the hawk takes all of the resource for themself. When two hawks meet a ferocious fight over the resource occurs between them, the victor takes the resource and the loser, not only receives nothing, but incurs a cost, c, of injury where $c > v > 0$. We assume all hawks are equally likely to win/lose this contest, so on average a hawk gets a payoff of $(v - c)/2 < 0$ when meeting another hawk. According to this, the normal form of the game is shown in the figure below where an animal A meets an animal B.

a). Find all equilibria in the game.

b). What could any mixed equilibria found in (a) represent contextually in this game?

c). If we were to set $c = v$, what game that we have seen does this become equivalent to? What about if $c < v$?

d). ($\diamond$) Look up the Bourgeois variant of Hawks and Doves. You might also like to investigate the concept of evolutionary stable strategies and replicator dynamics in this context.

		B	
		H	D
A	H	$\frac{V-C}{2}, \frac{V-C}{2}$	$V, 0$
	D	$0, V$	$\frac{V}{2}, \frac{V}{2}$

Figure 2: Hawks and Doves.

4). Determine **all** equilibria in the following two-player games:

a). A variant of the **inspection game** (see section 3.2, page 36 of the lecture notes) where the inspectee gets no gain from acting illegally.

		B	
		b_1	b_2
A	a_1	$0, 0$	$-10, 0$
	a_2	$-1, 0$	$-6, -90$

b). The **defence of Rome** (see section 2.3, page 18 of the lecture notes).

c). The 2×3 game below.

			B	
		b_1	b_2	b_3
A	a_1	$0, 2$	$3, 1$	$2, 0$
	a_2	$1, 0$	$2, 1$	$0, 2$

5). Prove that in any equilibrium of a two-player game that is **not** degenerate, both players use mixed strategies that mix over the same number of pure strategies.

6). In a two-player game let $\alpha, \hat{\alpha} \in \mathbb{A}_S$ and $\beta \in \mathbb{B}_S$. Suppose that (α, β) and $(\hat{\alpha}, \beta)$ are both equilibria of the game. Prove that $(k\alpha + (1-k)\hat{\alpha}, \beta)$ is also an equilibrium of the game for any $k \in [0, 1]$.

7). $(\star)(\diamond)$ Here's a very pretty result that I would've loved to derive in lectures had we more time to go into extra theory:

Theorem: A finite two-player non-degenerate game has an **odd** number of equilibria.

Prove this result.

To prove this you need some tools that we haven't developed in our course: the idea of labelling in best-response diagrams and the Lemke-Howson algorithm. If this result intrigues you and you want to learn some theory and see a proof then one resource is pages $223 - 235$ of the book 'Game Theory Basics'.