Intro to Game Theory: Problem Set 4 Solutions:

1).

a).

		B	
A	b_1	b_2	b_3
a_1	3	$\boxed{-4}$	1
a_2	2	6	$\boxed{0}$
a_3	5	4	$\boxed{3}$

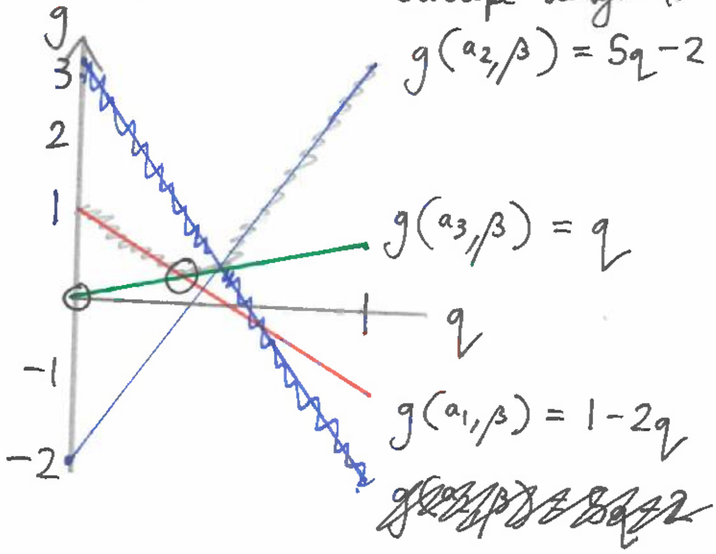
(a_3, b_3) is a pure strategy equilibrium, so these are optimal strategies for our players and the game has value: $v = 3$.

b).

		B
A	b_1	b_2
a_1	$\boxed{-1}$	1
a_2	3	$\boxed{-2}$
a_3	1	$\boxed{0}$

$\beta = (q, 1-q)$

• no pure strategy equilibria. Let's draw an upper-envelope diagram.



Min-max strategy for B is to play $1-2q = q \iff q = \frac{1}{3}$, or $\hat{\beta} = \left(\frac{1}{3}, \frac{2}{3}\right)$

A will then mix over a_1 and a_3, and to guarantee B's indifference we need:

$g(\alpha, b_1) = -p + 1 - p = 1 - 2p$ so let $\alpha = (p, 0, 1-p)$.

$g(\alpha, b_2) = p$, to be equal.

$\Rightarrow 1 - 2p = p$, or $p = \frac{1}{3}$, or $\hat{\alpha} = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$.

So $(\hat{\alpha}, \hat{\beta})$ are a pair of optimal strategies for A and B and the value of the game is $v = \frac{1}{3}$.

c).

A \ B	b_1	b_2	b_3
a_1	$\boxed{1}$	4	3
a_2	3	$\boxed{2}$	4
a_3	5	$\boxed{1}$	4

- No pure strategy equilibria.
- Observe that b_3 is dominated by $\beta = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$,

Since:
$$g(a_1, \beta) = \frac{5}{2} < 3 = g(a_1, b_3)$$
$$g(a_2, \beta) = \frac{5}{2} < 4 = g(a_2, b_3)$$
$$g(a_3, \beta) = 3 < 4 = g(a_3, b_3),$$

So we can <u>eliminate</u> b_3.

A \ B	b_1	b_2
a_1	$\boxed{1}$	4
a_2	3	$\boxed{2}$
a_3	5	$\boxed{1}$

- Now observe that a_2 is weakly dominated by $\alpha = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, since:

$$g(\alpha, b_1) = 3 = g(a_2, b_1)$$
$$g(\alpha, b_2) = \frac{5}{2} > 2 = g(a_2, b_2),$$

So we can freely <u>eliminate</u> a_2 when looking for <u>an</u> optimal pair of strategies and the value of the game.

- We then find that $\hat{\alpha} = \left(\frac{4}{7}, 0, \frac{3}{7}\right)$, $\hat{\beta} = \left(\frac{3}{7}, \frac{4}{7}, 0\right)$ give a pair of max-min/min-max strategies for the game. The value of the game is $g(\hat{\alpha}, \hat{\beta}) = \frac{19}{7}$.

d). It is clear there is no pure strategy equilibrium or any dominated strategies from the symmetrical payoff structure.

Instead, we guess, owing to this symmetry, that $\hat{\alpha} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is an equaliser strategy for A. This is easy to check: $g(\hat{\alpha}, b_1) = \frac{10}{4}$, and clearly (since each column contains a 1, 2, 3 and 4), all other pure strategies of B give the same payoff. Similarly, $\hat{\beta} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is an equaliser strategy for B. Thus $(\hat{\alpha}, \hat{\beta})$ forms an equilibrium of the game and the value of the game is $v = \frac{5}{2}$.

e).

		B		
		b_1	b_2	b_3
	a_1	x	0	0
A	a_2	0	y	0
	a_3	0	0	z

- First note that, if any of $x, y, z = 0$, then we get a pure strategy equilibrium at this point, where the value to A becomes 0.

* (see point below!!)

- Now assume $x, y, z \neq 0$. Owing to the symmetrical structure of the payoffs, we seek a pair of equaliser strategies for the players in the form:

$\hat{\alpha} = \hat{\beta} = (p, q, 1-p-q)$. We find:

$$\left.\begin{aligned} g(\hat{\alpha}, b_1) &= px \\ g(\hat{\alpha}, b_2) &= qy \\ g(\hat{\alpha}, b_3) &= z - pz - qz \end{aligned}\right\}$$

setting these equal gives: $px = qy$, or: $q = p\frac{x}{y}$

then:

$$px = z - pz - p\frac{xz}{y}$$

$$\Rightarrow \quad p\left(x + \frac{xz}{y} + z\right) = z$$

$$\Rightarrow \quad p = \frac{yz}{xy + xz + yz}$$

Thus we find: $\hat{\alpha} = \hat{\beta} = \left(\frac{yz}{D}, \frac{xz}{D}, \frac{xy}{D}\right)$, where $D = xy + xz + yz$, gives a pair of equaliser strategies for the players. The value of the game is $V = \frac{xyz}{D}$.

* • Next assume that x, y, z are not all the same sign. But then, by deleting weakly dominated strategies for both players, we reach a game with a pure strategy equilibrium at a 0 and value; $V = 0$.

Thus, assume that x, y, z all have the same sign and ... (return to above) f). Let's consider the same game with 1 added to every payoff, giving:

		B		
		b_1	b_2	b_3
A	a_1	$x+1$	0	0
	a_2	0	$y+1$	0
	a_3	0	0	$z+1$

• Now the game is exactly the same as in e), where $x \mapsto x+1$, $y \mapsto y+1$, $z \mapsto z+1$, so the results found in e). hold for this game. Hence, in our original game, the same results hold but we take away 1 from the value of the game.

2).

a). If A bids a sweets and B bids b sweets, then:

$$g_A(a,b) = \begin{cases} 4-a, & a>b \\ b, & a<b \\ 2, & a=b \end{cases}$$

Similarly:

$$g_B(a,b) = \begin{cases} a, & a>b \\ 4-b, & b>a \\ 2, & a=b. \end{cases}$$

Thus we have a constant sum game: $\underline{\underline{g_A(a,b)+g_B(a,b) = 4}}$.

(we can remove 2 from both players payoffs to get a zero-sum game).

Let $a_k = b_k =$ bid k sweets $(k=0,1,2,3,4)$, then we get:

		B				
		b_0	b_1	b_2	b_3	b_4
A	a_0	0	−1	0	1	2
	a_1	1	0	0	1	2
	a_2	0	0	0	1	2
	a_3	−1	−1	−1	0	2
	a_4	−2	−2	−2	−2	0

Either by iteratively deleting dominated / weakly-dominated strategies, or noting that (a_1, b_1), (a_1, b_2), (a_2, b_1) and (a_2, b_2) are pure strategy equilibria, we see that the value of the moderated game is 0 to each player, meaning the value of the <u>original game is 2</u> to <u>both players</u>.
(bidding 1 or 2 is optimal).

b). In this case we can check that bidding <u>2 sweets</u> is a best response for both players mutually: there is no incentive to bid less, since then your opponent pays you 2 sweets (what you would recieve if you had bid 2), and there is no incentive to bid more, since then you would stand to gain less than 2. So the value of the game is 2 to both players and they should bid 2.

c). (◇)

3). Consider the game G:

$$\begin{pmatrix} \boxed{1} & 2 & 2 \\ \boxed{0} & 2 & 1 \\ \boxed{0} & \boxed{0} & 3 \end{pmatrix}.$$

This has a pure-strategy equilibrium at (a_1, b_1) (top left entry). But the game G^T:

$$\begin{pmatrix} 1 & \boxed{0} & \boxed{0} \\ 2 & 2 & \boxed{0} \\ 2 & \boxed{1} & 3 \end{pmatrix},$$

has <u>no</u> pure-strategy equilibria.

So this is <u>not</u> true.

4).

a).

		B	
	R	P	S
A R	0	−1	1
A P	1	0	−1
A S	−1	1	0

- No pure-strategy equilibria.
- In any subgame where one pure strategy is removed for one player, e.g. removes S for A, then one strategy for the opponent becomes dominated (here if A doesn't play S then R is dominated by P for B), so the game becomes a 2×2 game with no pure equilibria. ~~the~~ Any mixed equilibrium in this subgame will be inferior to the originally removed strategy (for us S for A dominates P if B is not playing R) so these equilibria will not be equilibria of the full game. This means only an equilibrium where both players mix over all three pure strategies is possible.

There can therefore only be one equilibrium of this form (owing to the amount of free variables, 2, to satisfy the two indifference equations for each player). One can check that $\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)$ is the equilibrium. The game has value 0; it is a fair game.

b). Let $a_k = b_k$ = the player holds up k fingers; $k = 1, 2, 3, 4, 5$. Then:

				B		
		b_1	b_2	b_3	b_4	b_5
	a_1	0	1	−1	−1	−1
	a_2	−1	0	1	−1	−1
A	a_3	1	−1	0	1	−1
	a_4	1	1	−1	0	1
	a_5	1	1	1	−1	0

c). A zero-sum game has a unique value. The game is symmetric, so when changing the players and changing the signs of the payoffs its value changes sign but must be the same, which is only possible if the value is <u>zero</u>.

Alternatively, one can think of the value as the max-min payoff to A and min-max cost to B. If the value was positive, then player A would get a positive payoff and player B could adopt player A's strategy and also get a positive amount, which is not possible. Similarly if the value was negative. So the value must be zero.

d). We can eliminate weakly dominated strategies and not change the value of the game, so we delete a_1 and a_2 (weakly dominated by a_5) and we delete b_1 and b_2 (weakly dominated by b_5). This leaves us with the game: $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, which is exactly the same as the game Rock-Paper-Scissors, so ~~the~~ an optimal strategy is for both players to play $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in this game, meaning that

$\alpha = \beta = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ forms an equilibrium in the fingers game.

5). Without loss of generality, assume $V \leq W$ (if $W < V$ then we can relabel $b_1 \mapsto b_3$, $b_2 \mapsto b_4$). This means that b_1 weakly dominates or is payoff equivalent to b_3 and b_2 weakly dominates or is payoff equivalent to b_4, so we can delete b_3 and b_4. We are left with the game:

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$$\begin{array}{cc|c|c} & & \multicolumn{2}{c}{B} \\ & & b_1 & b_2 \\ \hline & a_1 & V & V \\ \hline A & a_2 & x & y \\ \hline & a_3 & x & z \end{array}$$

Without loss of generality, let's assume $z \leqslant y$ (if $z > y$ then we can relabel $a_2 \mapsto a_3$). Then a_3 is weakly dominated by or payoff equivalent to a_2, so we can delete a_3. We get:

$$\begin{array}{cc|c|c} & & \multicolumn{2}{c}{B} \\ & & b_1 & b_2 \\ \hline & a_1 & V & V \\ \hline A & a_2 & x & y \end{array}$$

Without loss of generality, let's assume $x \leqslant y$ (if $x > y$ we can relabel $b_1 \mapsto b_2$). Then b_1 weakly dominates or is payoff equivalent to b_2, so we can delete b_2. This leaves us with the game:

$$\begin{array}{cc|c} & & B \\ & & b_1 \\ \hline & a_1 & V \\ \hline A & a_2 & x \end{array},$$

where a pure strategy equilibrium is guaranteed (at the larger of V or x, or both if they are equal).

6).

a). $\underline{1 \leqslant x \leqslant 3}$:

		B		
		b_1	b_2	b_3
A	a_1	$\boxed{1}$	2	3
	a_2	$\boxed{2}$	3	$\boxed{2}$
	a_3	3	$\boxed{2}$	$\boxed{x}$

if $x \leqslant 2$

if $x = 3$

if $x \geqslant 2$

– So no pure strategy equilibria.

Owing to the symmetrical structure of payoffs, seek a pair of equaliser strategies of the form: $\alpha = \beta = (p, q, 1-p-q)$. These are ES's if:

$$g(\alpha, b_1) = p + 2q + 3(1-p-q) = 3 - 2p - q$$

$$= g(\alpha, b_2) = 2p + 3q + 2(1-p-q) = 2 + q$$

$$= g(\alpha, b_3) = 3p + 2q + x(1-p-q) = x + (3-x)p + (2-x)q.$$

Solving these gives: $p = \dfrac{3-x}{4}$, $q = \dfrac{x-1}{4}$, which if $1 \leqslant x \leqslant 3$ these form valid probabilities. Hence we get:

$$\hat{\alpha} = \hat{\beta} = \left(\frac{3-x}{4}, \frac{x-1}{4}, \frac{1}{2}\right)$$ as a pair of optimal strategies giving the game value: $$\underline{\underline{g(\hat{\alpha}, \hat{\beta}) = \frac{x+7}{4}}}.$$

b). (i). $x > 3$. In this case b_1 weakly dominates b_3, so we can delete b_3.
Then a_2 dominates a_1, so delete a_1. We are left with:

		B	
		b_1	b_2
A	a_2	2	3
	a_3	3	2

One finds that $\hat{\alpha}_s = \hat{\beta}_s = \left(\frac{1}{2}, \frac{1}{2}\right)$ gives an equilibria in this game, hence:
$\hat{\alpha} = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\hat{\beta} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ is an equilibrium in the original game.
The game has value $g(\hat{\alpha}, \hat{\beta}) = \frac{5}{2}$.

(ii). $x < 1$. In this case b_2 is weakly dominated by $\beta = \frac{1}{2} b_1 + \frac{1}{2} b_3$, so we can delete b_2. This gives:

		B	
		b_1	b_3
	a_1	1	3
A	a_2	2	2
	a_3	3	x

a_2 is therefore maximin as it's an equaliser strategy for A. Letting B play $\beta = (q, 0, 1-q)$, then β is minimax if:

$$g(a_1, \beta) = q + 3(1-q) \leq 2 \text{, and}$$
$$g(a_3, \beta) = 3q + x(1-q) \leq 2 \text{, i.e. if: } \frac{1}{2} \leq q \leq \frac{2-x}{3-x}.$$

So: $(a_2, (q, 0, 1-q))$, with $\frac{1}{2} \leq q \leq \frac{2-x}{3-x}$, give a pair of optimal strategies.
The game has value; $v = 2$.

7). Denote a_k to mean A defends k and b_k to mean B attacks k.

a). $m = 6$:

A \ B	b_1	b_2	b_3	b_4	b_5	b_6
a_1	0	0	-1	-1	-1	-1
a_2	0	0	0	-1	-1	-1
a_3	-1	0	0	0	-1	-1
a_4	-1	-1	0	0	0	-1
a_5	-1	-1	-1	0	0	0
a_6	-1	-1	-1	-1	0	0

- We delete a_1; it's weakly dominated by a_2, and we delete a_6; it's weakly dominated by a_5.

We delete b_2; it's weakly dominated by b_1, and we delete b_5; it's weakly dominated by b_6.

- We get:

A \ B	b_1	b_3	b_4	b_6
a_2	0	0	-1	-1
a_3	-1	0	0	-1
a_4	-1	0	0	-1
a_5	-1	-1	0	0

- We delete b_3; weakly dominated by b_1, and we delete b_4; weakly dominated by b_6. We can delete one of a_3 and a_4 as these are payoff equivalent.
- Upon doing this the remainder of a_3/a_4 is weakly dominated by a_2 or a_5 so we can delete the other. This leaves us with:

	b_1	b_6
a_2	0	-1
a_5	-1	0

• We can find that $\hat{\alpha}_S = \hat{\beta}_S = \left(\frac{1}{2}, \frac{1}{2}\right)$ gives an equilibrium in this game, hence:
$\hat{\alpha} = \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right)$, $\hat{\beta} = \left(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}\right)$ form an equilibrium in the full game. The value is $\underline{\underline{g(\hat{\alpha}, \hat{\beta}) = -\frac{1}{2}}}$.

b). $\underline{m=8}$: Following a similar procedure to a), we can reduce the game to:
(or have a_4, b_4 instead of 5)

	b_1	b_5	b_8
a_2	0	−1	−1
a_5	−1	0	−1
a_7	−1	−1	0

One can check that $\hat{\alpha}_S = \hat{\beta}_S = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ gives an equilibrium in this game.
So:

$$\hat{\alpha} = \left(0, \tfrac{1}{3}, 0, 0, \tfrac{1}{3}, 0, \tfrac{1}{3}, 0\right)$$
$$\hat{\beta} = \left(\tfrac{1}{3}, 0, 0, 0, \tfrac{1}{3}, 0, 0, \tfrac{1}{3}\right)$$

forms an equilibrium here.
This game has value $\underline{\underline{V = -\frac{2}{3}}}$.

c). (◇)

8).

a). (α, β) in equilibrium means that:

$$g(\alpha', \beta) \overset{①}{\leqslant} g(\alpha, \beta) \overset{②}{\leqslant} g(\alpha, \beta'), \quad \text{for all } \alpha' \in \mathbb{A}_S, \ \beta' \in \mathbb{B}_S.$$

Similarly (γ, δ) in equilibrium gives:

$$g(\gamma', \delta) \overset{③}{\leqslant} g(\gamma, \delta) \overset{④}{\leqslant} g(\gamma, \delta'), \quad \text{for all } \gamma' \in \mathbb{A}_S, \ \delta' \in \mathbb{B}_S.$$

This means we have:

$$g(\alpha,\beta) \underset{\text{by } ②}{\leqslant} g(\alpha,\delta) \underset{\text{by } ③}{\leqslant} g(\gamma,\delta) \underset{\text{by } ④}{\leqslant} g(\gamma,\beta) \underset{\text{by } ①}{\leqslant} g(\alpha,\beta).$$

So all inequalities must be equalities; giving:

$$g(\alpha,\beta) = g(\alpha,\delta) = g(\gamma,\beta) = g(\gamma,\delta).$$

$\square$

b). For any $\lambda \in \mathbb{A}_S, \mu \in \mathbb{B}_S$ we have:

$$g(\lambda,\delta) \underset{\text{by } ③}{\leqslant} g(\gamma,\delta) \underset{\text{by a).}}{=} g(\alpha,\beta) \underset{\text{by } ②}{\leqslant} g(\alpha,\mu),$$

which means that (α,δ) are in equilibrium.

A similar argument shows (γ,β) are in equilibrium.

$\square$

9).

a). Let the payoffs to the maximising row player in this zero-sum game be a_{ij} for $i = 1, 2$ and $j = 1, 2, \dots, n$. Because we know that a_{11} is an equilibrium and the best responses are unique, we have:

$$a_{11} > a_{21}, \quad a_{11} < a_{1j} \text{ for } j = 2, 3, \dots, n.$$

Therefore, any column j with $a_{21} < a_{2j}$ is strictly dominated by column 1 and can be deleted.

After this, any column $j \neq 1$ fulfills:

$$a_{2j} \leqslant a_{21} < a_{11} < a_{1j},$$

but this means that row 1 strictly dominates row 2, so then row 2 can be deleted. Clearly then all columns other than 1 for the single row 1 can also be eliminated as being strictly dominated by column 1.

b). For a degenerate game the claim does <u>not</u> hold. For example:

		B	
		b_1	b_2
A	a_1	2	2
	a_2	1	0

is <u>not</u> dominance solvable but (a_1, b_1) is a pure strategy equilibria.

c). An example is given by:

			B	
		b_1	b_2	b_3
A	a_1	5	6	7
	a_2	4	8	1
	a_3	3	2	9

10). (◇)

11). (☆)(◇)