Problem Set 4

1). Find optimal strategies for both players and the value of the game in each of the two-player zero-sum games below ($x, y, z \in \mathbb{R}$):

a).

A / B	b_1	b_2	b_3
a_1	3	−4	1
a_2	2	6	0
a_3	5	4	3

b).

A / B	b_1	b_2
a_1	−1	1
a_2	3	−2
a_3	1	0

c).

A / B	b_1	b_2	b_3
a_1	1	4	3
a_2	3	2	4
a_3	5	1	4

d).

A / B	b_1	b_2	b_3	b_4
a_1	1	2	3	4
a_2	2	3	4	1
a_3	3	4	1	2
a_4	4	1	2	3

e).

A / B	b_1	b_2	b_3
a_1	x	0	0
a_2	0	y	0
a_3	0	0	z

f).

A / B	b_1	b_2	b_3
a_1	x	−1	−1
a_2	−1	y	−1
a_3	−1	−1	z

2). **The Inheritance Game:** Two people together inherit 4 sweets. They each make a sealed bid (in whole numbers of sweets) for outright ownership of these 4 sweets. When the two sealed bids are opened, the higher bidder inherits the 4 sweets and pays the other person an amount of sweets equal to the higher bid. If both make an equal bid then a fair coin is tossed to determine who shall inherit the sweets and this person pays the other the amount of the (common) bid.

a). Find a solution to the game and the value of the game to each player.

b). Find a solution to the game if we allow for any real number denomination of sweets to be bid.

c). ($\diamond$) An interesting extension could be to factor in some uncertainty in the amount of sweets to be inherited. An extra player could also be included.

3). Suppose a finite two-player zero-sum game, G, has a pure strategy equilibrium, then does the game G^T (the game with normal form given by the transpose of G) necessarily have a pure strategy equilibrium?

4). a). Solve the game of Rock-Paper-Scissors (see chapter 2, page XX).

The Fingers Game: Consider the two-player simultaneous game where each player chooses an integer between 1 and 5 inclusive (in practice by raising some number of fingers up on their hand), and the higher number wins, except when it is just one larger than the lower number, in which case the lower number wins. Equal numbers give a draw. So, for example, 4 beats 1, 2 and 5, but loses against 3 and draws with 4.

b). Write down the matrix of payoffs to the row player of this zero-sum game. Let a winning player get a payoff of 1, a losing player get -1 and a draw give 0.

c). Without performing any calculations; what is the value of the game and why?

d). Find a solution of this game.

5). Show that, for any real numbers v, w, x, y and z, the two-player zero-sum game given in the figure below **always** has a pure strategy equilibrium.

		B			
		b_1	b_2	b_3	b_4
	a_1	v	v	w	w
A	a_2	x	y	x	y
	a_3	x	z	x	z

6). Consider the zero-sum game below where $x \in \mathbb{R}$.

			B	
		b_1	b_2	b_3
	a_1	1	2	3
A	a_2	2	3	2
	a_3	3	2	x

a). Show that, when $1 \leqslant x \leqslant 3$, the game has value $\frac{x+7}{4}$ and find optimal strategies for A and B.

b). Find a solution to the game (and its value), when:

(i). $x > 3$;

(ii). $x < 1$.

7). An army, A, is trying to defend m targets against an enemy, B. The targets are arranged in a line, numbered 1 to m from left to right, and only one target and any immediately next to it can be defended. B only has sufficient resources to attack a single target. If B attacks a defended target there is no loss to either side but if B attacks an undefended target then A loses the target to B. All targets are equally valuable to both sides.
Find good strategies for A and B and the value of the game to A when:

a). $m = 6$;

b). $m = 8$;

c). ($\diamond$) Other values of m.

8). In a two-player zero-sum game, where $\alpha, \gamma \in \mathbb{A}_S$ and $\beta, \delta \in \mathbb{B}_S$, suppose that (α, β) and (γ, δ) are both equilibria. Prove that:

a). $g(\alpha, \beta) = g(\alpha, \delta) = g(\gamma, \beta) = g(\gamma, \delta)$;

b). Both (α, δ) and (γ, β) are also equilibria.

9). a). Consider a $2 \times n$ zero-sum game where the best response to every pure strategy is unique. Suppose that (a_1, b_1) is a pure strategy equilibrium of the game. Show that this equilibrium can be found by iterated deletion of strictly dominated strategies.

b). Does (a) still hold if some pure strategy has more than one best response (and the game is therefore degenerate)?

c). Give an example of a 3×3 zero-sum game, where the best response to every pure strategy is unique, which has a pure strategy equilibrium that **cannot** be found by iterated deletion of strictly dominated strategies.

10). ($\diamond$) Consider a 2×2 two-player zero-sum game with real valued payoffs w, x, y and z, as shown below.

		B	
		b_1	b_2
A	a_1	w	x
	a_2	y	z

Assuming that the game has **no** pure strategy equilibria:

a). Show that player A should play

$$\alpha = \left(\frac{|y - z|}{|w - x| + |y - z|}, \frac{|w - x|}{|w - x| + |y - z|} \right).$$

b). What should player B play?

c). What is the value of the game?

11). ($\star$)($\diamond$) **The G^{-1} method for zero-sum games:** If both players have the same number of pure strategies in a two-player zero-sum game then the game, G, can be thought of as a square matrix. Assuming G^{-1} exists (the inverse matrix), we can use it to look for a pair of **equaliser strategies** for the players.

a). Denoting $\mathbf{1}$ as a column vector of ones, can you show that, provided the row sums and the column sums of G^{-1} have the same sign, a pair of equaliser strategies is given by

$$\alpha = \frac{\mathbf{1}^T G^{-1}}{\mathbf{1}^T G^{-1} \mathbf{1}}, \quad \beta = \frac{\mathbf{1}^T (G^{-1})^T}{\mathbf{1}^T (G^{-1})^T \mathbf{1}},$$

and the value of the game is then

$$v = \frac{1}{\mathbf{1}^T G^{-1} \mathbf{1}}.$$

b). Why do we need this restriction on the row and column sums to have the same sign?

c). Solve the game

$$G = \begin{bmatrix} 5 & 0 & 5 \\ 7 & -3 & -1 \\ 3 & 5 & 4 \end{bmatrix},$$

and show that it has value $v = \frac{185}{51}$ using the G^{-1} method outlined above.