Intro to Game Theory: Problem Set 5 Solutions

1).

a).

(i). Let's consider A's payoffs as if a zero-sum game with A as maximiser:

A \ B	b_1	b_2
a_1	$\boxed{1}$	3
a_2	$\boxed{0}$	3

- This game has a pure strategy equilibrium which means that the max-min strategy of player A is a_1, with max-min payoff $\underline{t_A = 1}$.

Now consider B's payoffs as if a zero-sum game with B as maximiser:

A \ B	b_1	b_2
a_1	$\boxed{-2}$	3
a_2	6	$\boxed{1}$

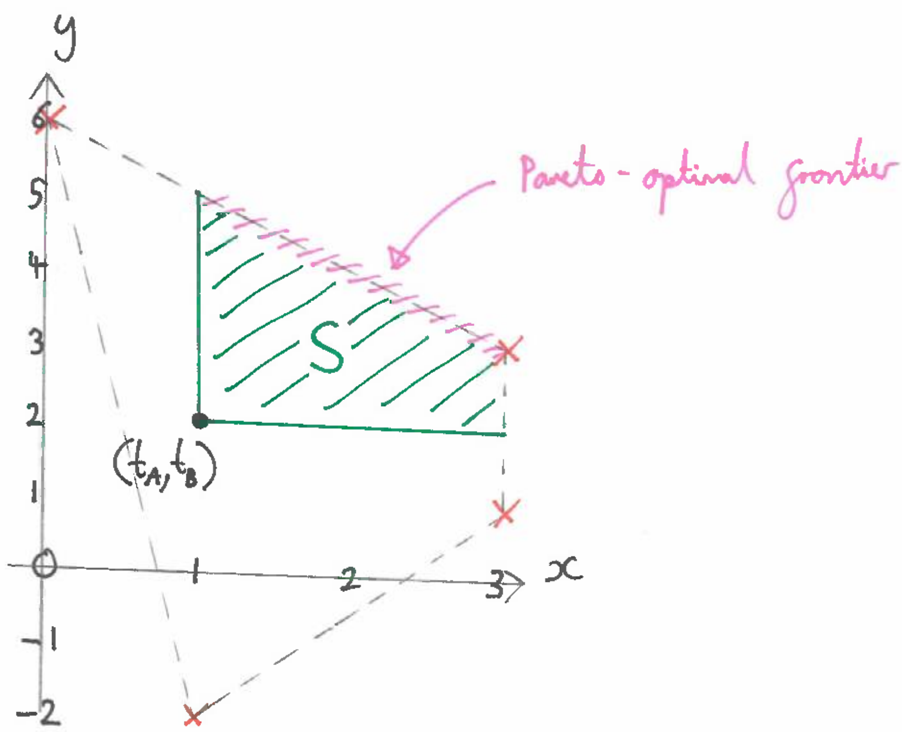
- No pure strategy equilibrium, we seek a mixed strategy equilibrium: one finds:

$$\left.\begin{aligned} g_B(a_1,\beta) &= 3-5q \\ g_B(a_2,\beta) &= 1+5q \end{aligned}\right\} \Rightarrow \underline{q = \frac{1}{5}} \text{ for indifference.}$$

So this means that $\hat{\beta} = \left(\frac{1}{5}, \frac{4}{5}\right)$ is a max-min strategy for B, with max-min payoff equal to $\underline{t_B = 2}$.

Threat point is $(1,2)$.

(ii).



(iii). We maximise the Nash product over the pareto-optimal frontier; which has equation $y = 6 - x$:

$$(x - t_A)(y - t_B) = (x-1)(y-2) = (x-1)(4-x)$$
$$= -x^2 + 5x - 4,$$

which is maximised when $\underline{\underline{x = \frac{5}{2}}}$, which $\underline{\text{is}}$ on the line segment, so when $x = \frac{5}{2}$ and $y = \frac{7}{2}$ we have the Nash bargaining solution.

(iv). The players can achieve this payoff pair $\left(\frac{5}{2}, \frac{7}{2}\right)$ by playing the joint strategy:
- choose (a_1, b_2) with probability $\frac{5}{6}$ and (a_2, b_1) with probability $\frac{1}{6}$.

b).

(i). Considering A's payoffs:

A \ B	b_1	b_2
a_1	$\boxed{2}$	4
a_2	3	$\boxed{1}$

- No pure strategy equilibrium, seek a mixed strategy equilibrium. $\alpha = (p, 1-p)$.

$$\left.\begin{aligned} g_A(\alpha, b_1) &= 3 - p \\ g_A(\alpha, b_2) &= 1 + 3p \end{aligned}\right\} \Rightarrow \underline{\underline{p = \frac{1}{2}}} \text{ for indifference.}$$

This means that $\hat{\alpha} = \left(\frac{1}{2}, \frac{1}{2}\right)$ is max-min for A with max-min payoff equal to $\underline{\underline{\frac{5}{2} = t_A}}$.

Now consider B's payoffs:

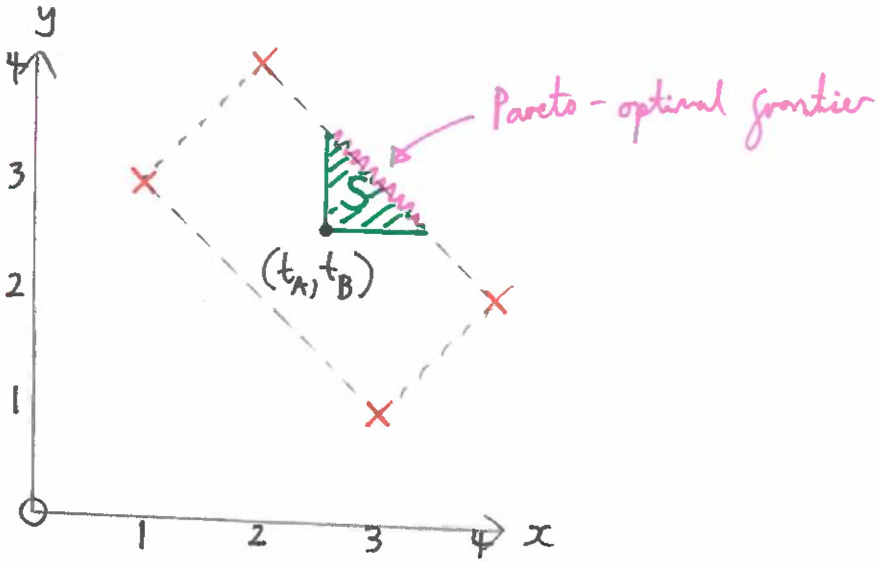
A \ B	b_1	b_2
a_1	4	$\boxed{2}$
a_2	$\boxed{1}$	3

- No pure strategy equilibrium. Seek a mixed strategy equilibrium. $\beta = (q, 1-q)$.

$$\left.\begin{aligned} g_B(a_1, \beta) &= 2 + 2q \\ g_B(a_2, \beta) &= 3 - 2q \end{aligned}\right\} \Rightarrow \underline{\underline{q = \frac{1}{4}}} \text{ for indifference.}$$

This means that $\hat{\beta} = \left(\frac{1}{4}, \frac{3}{4}\right)$ is max-min for B with max-min payoff equal to $t_B = \frac{5}{2}$. Threat point is $\left(\frac{5}{2}, \frac{5}{2}\right)$ for the game.

(ii).



(iii). By the symmetry, the Nash bargaining solution must lie on line $y = x$, so it is $x = 3,\ y = 3$.

(iv). Player A plays a_1 and player B plays b_1 or b_2 with equal probability.

c).

(i). A's payoffs:

		B: b_1	b_2
A	a_1	2	3
	a_2	0	2

• Pure strategy equilibrium which means that a_1 is a max-min strategy of A, with max-min payoff equal to $t_A = 2$.

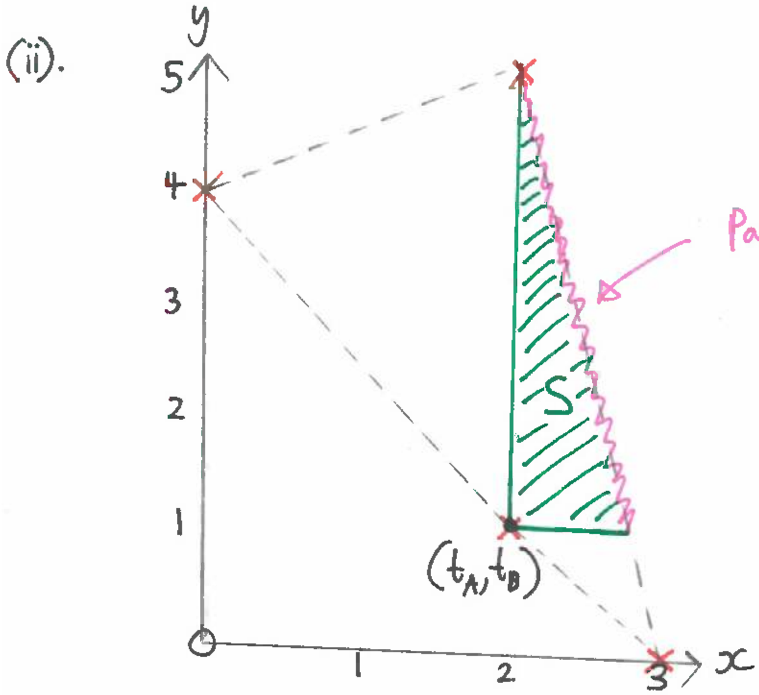
B's payoffs:

		B: b_1	b_2
A	a_1	1	0
	a_2	4	5

• Pure strategy equilibrium which means that b_1 is a max-min strategy of B, with max-min payoff equal to $t_B = 1$.

Threat point $(2, 1)$.

(ii).



iii). Maximise the Nash product over the pareto-optimal frontier; which has equation $y = 15 - 5x$:

$$(x - t_A)(y - t_B) = (x-2)(y-1) = (x-2)(14-5x)$$

$$= -5x^2 + 24x - 28,$$

which is maximised when $x = \frac{12}{5}$, which is on the line segment, so when $x = \frac{12}{5}$ and $y = 3$ we have the Nash bargaining solution.

iv). The players can implement this solution by playing the joint strategy:

- choose (a_2, b_2) with probability $\frac{3}{5}$ and (a_1, b_2) with probability $\frac{2}{5}$.

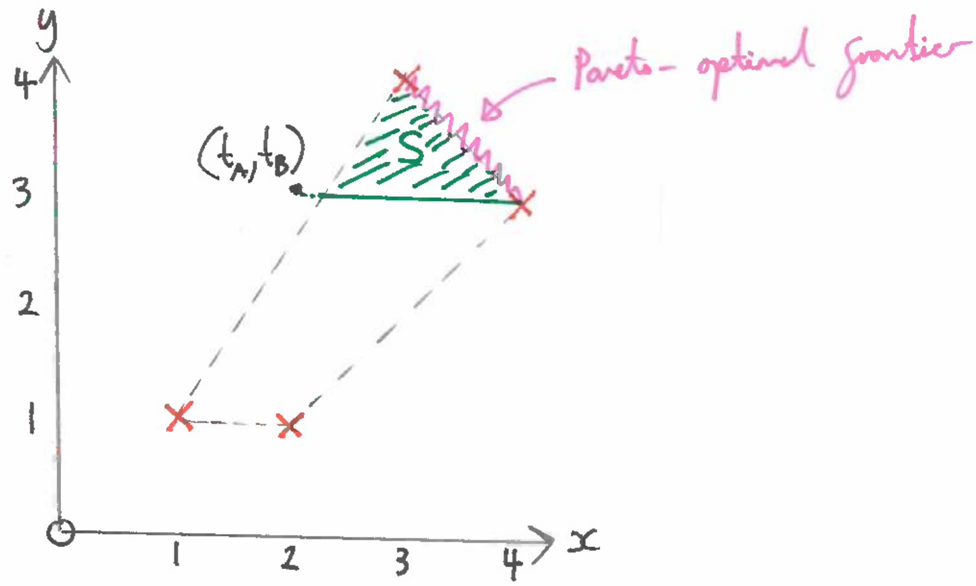
(or B plays b_2, A plays $(\frac{2}{5}, \frac{3}{5})$).

d).

(i). The max-min strategy of A is a_1 with payoff $t_A = 2$ and the max-min strategy of B is b_1 with payoff $t_B = 3$.

Threat point $(2, 3)$.

(ii).



(iii). Maximise Nash product over the pareto-optimal frontier; which has equation $y = 7 - x$:

$$(x - t_A)(y - t_B) = (x-2)(y-3) = (x-2)(4-x)$$
$$= -x^2 + 6x - 8,$$

which is maximised when $x = 3$, which is on the line segment, so when $x = 3$ and $y = 4$ we have the Nash bargaining solution.

(iv). The players play (a_1, b_1) with certainty to arrive at the Nash bargaining solution.

2). The pair (X, Y) maximises the Nash product $(x-0)(y-0)$ if $XY \geqslant xy$ for all $(x,y) \in S$.

Multiply this inequality by $ab > 0$; so then:

$$aXbY \geqslant axby \,, \text{ for all } (x,y) \in S,$$

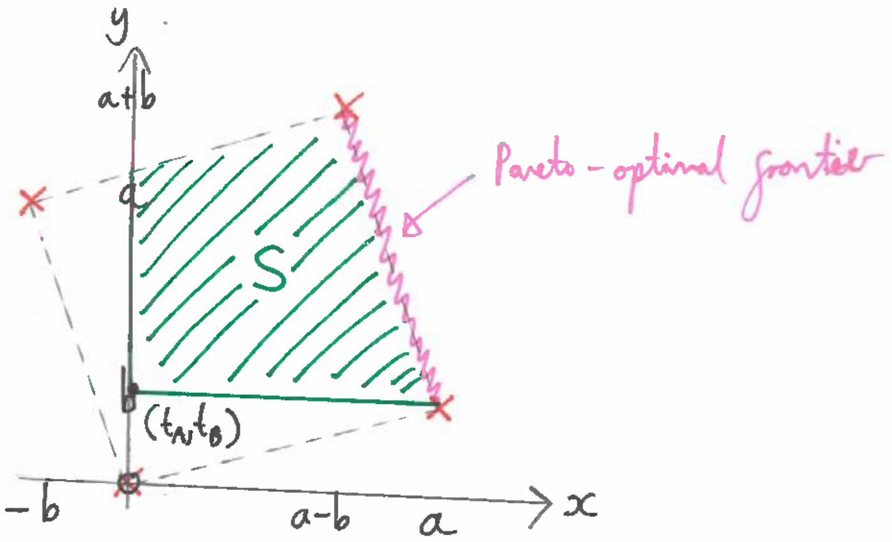
i.e. $$aXbY \geqslant x'y' \,, \text{ for all } (x', y') \in S'.$$

Thus the maximum Nash product in S' is indeed obtained for $(X', Y') = (aX, bY)$, as claimed. $\square$

3). Considering A's payoffs, a_2 is a max-min strategy for A with payoff 0, so that $t_A = 0$.

Considering B's payoffs, b_1 is a max-min strategy for B with payoff b, so that $t_B = b$.

We sketch the bargaining set:



We maximise: $(x-0)(y-b)$ on the pareto-optimal frontier where $y = -\frac{a}{b}x + \frac{a^2+b^2}{b}$; i.e: maximise:

$$x\left(-\frac{a}{b}x + \frac{a^2}{b}\right) = -\frac{a}{b}x^2 + \frac{a^2}{b}x,$$

which occurs when $\underline{\underline{x = \frac{a}{2}}}$, $\underline{\underline{y = \frac{a^2+2b^2}{2b}}}$.

• However, this only lies <u>in</u> the bargaining set if $\frac{a}{2} > a-b$, i.e. if $b > \frac{a}{2}$. Otherwise the Nash bargaining solution lies at $\underline{\underline{(a-b, a+b)}}$.

4).

(i). A's payoffs:

A \ B	b_1	b_2
a_1	$\boxed{1}$	4
a_2	3	$\boxed{1}$
a_3	4	$\boxed{2}$

- no pure strategy equilibria.
- a_2 is strictly dominated by a_3, so can be deleted.
- In the remaining 2×2 game we can seek an ~~mixed~~ equaliser strategy for A. We find that $\alpha = \left(\frac{2}{5}, 0, \frac{3}{5}\right)$ is max-min for A, giving payoffs $\frac{14}{5}$, so that $t_A = \frac{14}{5}$.

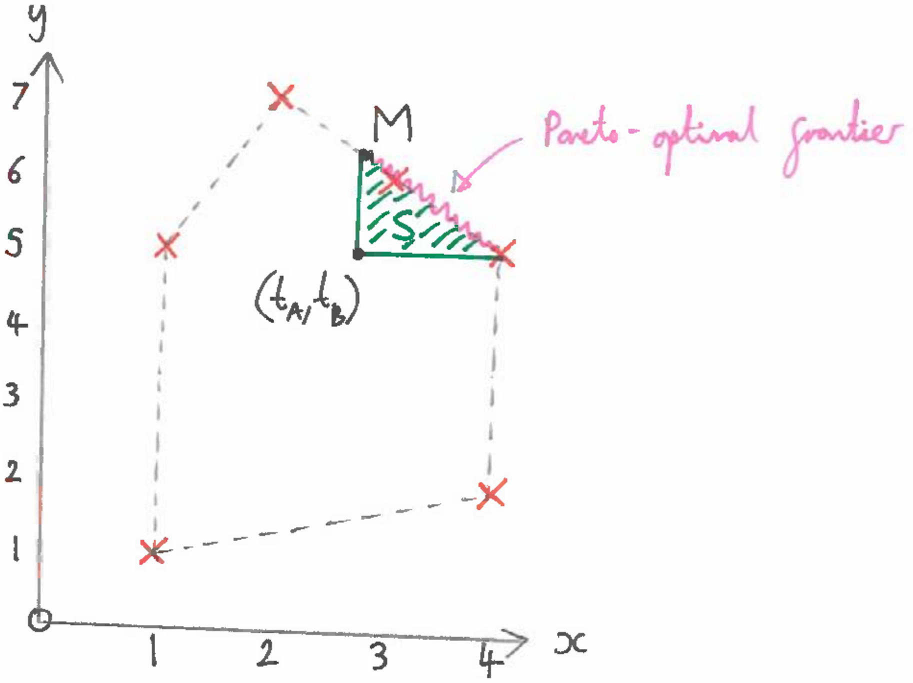
B's payoffs

A \ B	b_1	b_2
a_1	$\boxed{5}$	2
a_2	6	$\boxed{1}$
a_3	$\boxed{5}$	7

- Pure strategy equilibrium at (a_1, b_1), so then b_1 is a max-min strategy for B giving payoff 5, so then $t_B = 5$.

The threat point is $\left(\frac{14}{5}, 5\right)$.

(ii).



(iii). B cannot expect to get more than the y-coordinate of the point marked M on the diagram in (ii); this lies on the line $y = -x + 9$ when $x = \frac{14}{5}$, i.e. B cannot expect to get more than $9 - \frac{14}{5} = \frac{31}{5}$.

(e)

(iv). On the pareto-optimal frontier, $y = 9 - x$, and we maximise the Nash product, given by:

$$\left(x - \frac{14}{5}\right)(y - 5) = \left(x - \frac{14}{5}\right)(4 - x)$$
$$= -x^2 + \frac{34}{5}x - \frac{56}{5},$$

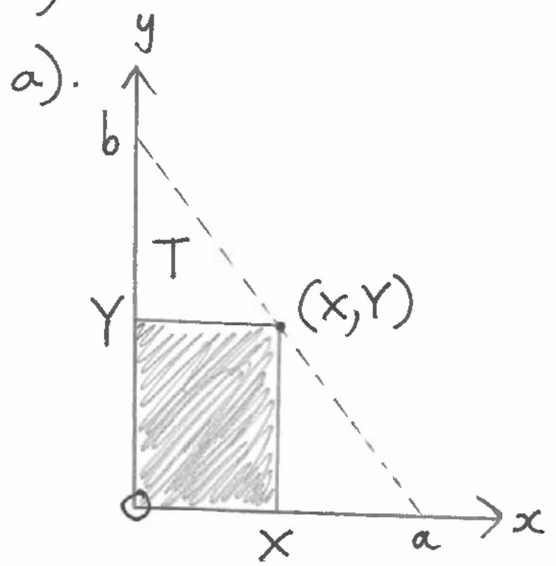
which is maximised when $\underline{\underline{x = \frac{17}{5}}}$ and $\underline{\underline{y = \frac{28}{5}}}$. Since this point lies <u>in</u> the bargaining set it gives the Nash bargaining solution.

(v). As there are three points $(2,7)$, $(3,6)$ and $(4,5)$ that lie colinear along the pareto-optimal frontier's line. in this game their are infinitely many possibilities for this.
One possibility is for the players to play:

- A plays a_3, B plays $\left(\frac{7}{10}, \frac{3}{10}\right)$, or, they could play:
- $\frac{3}{5}(a_2, b_1) + \frac{2}{5}(a_3, b_1)$, or a mixture of these appropriately weighted.

5).

a).



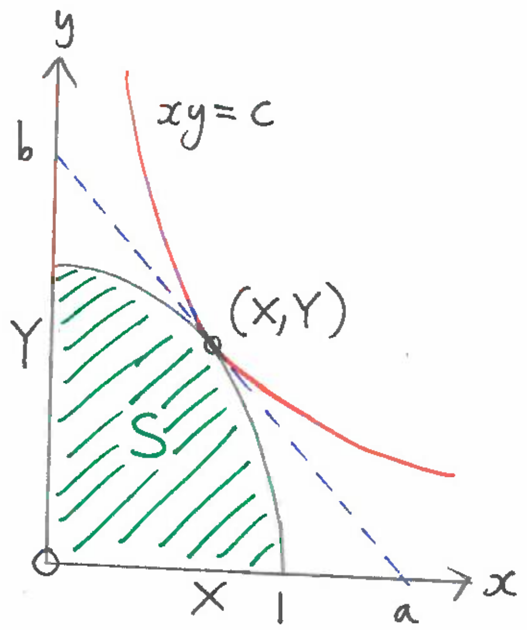
The product XY is the area of the rectangle inscribed in T as shown in the diagram. The maximum value of xy for $(x,y) \in T$ clearly results when (x,y) is on the line that joins $(0,b)$ and $(a,0)$; dashed in the diagram (due to pareto-optimality in essence!) with equation: $y = b - \frac{b}{a}x$.

Now consider the derivative of $xy = x\left(b - \frac{b}{a}x\right)$ which is $b - 2\frac{b}{a}x$, which is zero when $x = \frac{a}{2}$.

This is a maximum of the function because the second derivative is negative. Hence XY is maximal for $X = \frac{a}{2}$ and $Y = \frac{b}{2}$ as claimed.

If $S \subseteq T$, then the maximum of xy for $(x,y) \in S$ is clearly at most the maximum of xy for $(x,y) \in T$, and therefore attained for $(x,y) = (X,Y)$ if $(X,Y) \in S$. $\square$

b).



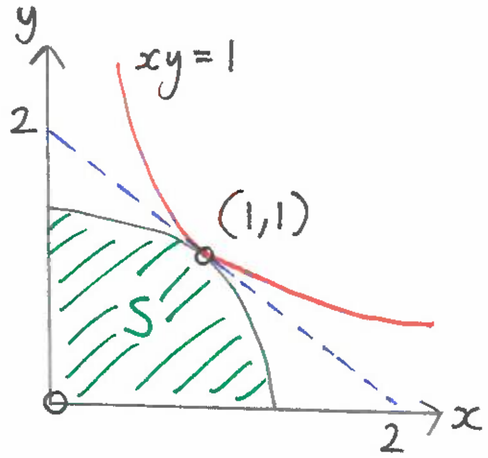
Let (X,Y) be the bargaining solution for set S. It is on the pareto-optimal frontier, so $Y = f(X)$. Let $c = XY$ and consider the hyperbola $\{(x,y) : xy = c\}$, which intersects S at the unique point (X,Y) by the uniqueness of the Nash bargaining solution.

Because (X,Y) maximises the Nash product, c is the maximal value of $c = xy$ for $(x,y) \in S$. The function $y = \frac{c}{x}$ is differentiable with derivative $\frac{-c}{x^2}$, which for $x = X$ is equal to $\frac{-Y}{X}$.

Consider now the line through (X,Y) with slope $\frac{-Y}{X}$, which is the tangent to the hyperbola at (X,Y), illustrated by a dashed blue line in the diagram.

This line intersects the x-axis at the point $(a,0)=(2X,0)$ and the y-axis at $(0,b)=(0,2Y)$, and defines a triangle, T, with vertices $(0,0)$, $(a,0)$, and $(0,b)$.

If we now re-scale the axes by replacing x with $\frac{x}{X}$ and y with $\frac{y}{Y}$ then we obtain the diagram below where T is now the triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$.



In this case we can show $S \subseteq T$ exactly analogously to how we did in the proof of Nash's bargaining solution, so this also applies to the previous figure, before we re-scaled the axes.

Hence the blue-dashed line is also a tangent line to the set S, as claimed.

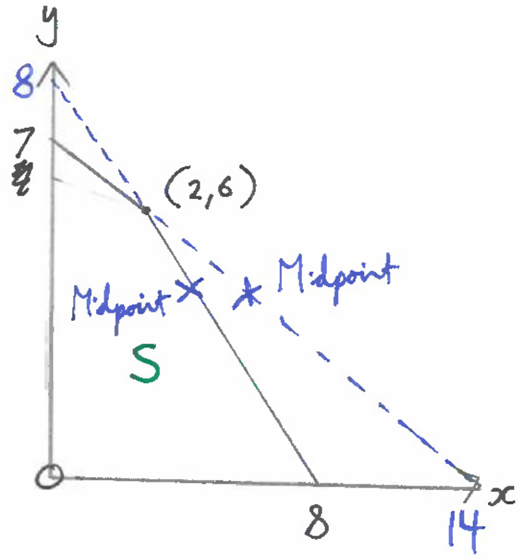
We now show uniqueness. Suppose that a tangent of S touches S at the point (X,Y) and has the slope $-\frac{Y}{X}$, intersects the x-axis at $(a,0)$ and intersects the y-axis at $(0,b)$ as in our first figure. The slope $-\frac{Y}{X}$ of the tangent implies that $a=2X$ and $b=2Y$. Being a tangent to the set S means that S is a subset of the triangle T with vertices $(0,0)$, $(a,0)$ and $(0,b)$.

By part (a), XY is the maximum value of xy for $(x,y)\in S$. This shows that (X,Y) is unique.

Finally if f is differentiable then the tangent to S is unique at every point. Now the bargaining solution maximises the Nash product $x f(x)$ on the pareto-optimal frontier. This requires the derivative with respect to x to be zero, i.e. $f(x) + x f'(x) = 0$, or $f'(x) = -\frac{f(x)}{x}$, so X has to solve this equation, as claimed.

c). With the help of part (a) we can more easily determine the Nash bargaining solution. We find a tangent to S with endpoints $(a,0)$ and $(0,b)$ so that the midpoint (X,Y) of that tangent belongs to S (part (b) asserts that this tangent will always exist). In the cases where S is built from line segments, we need only check the above for each separate line segment.

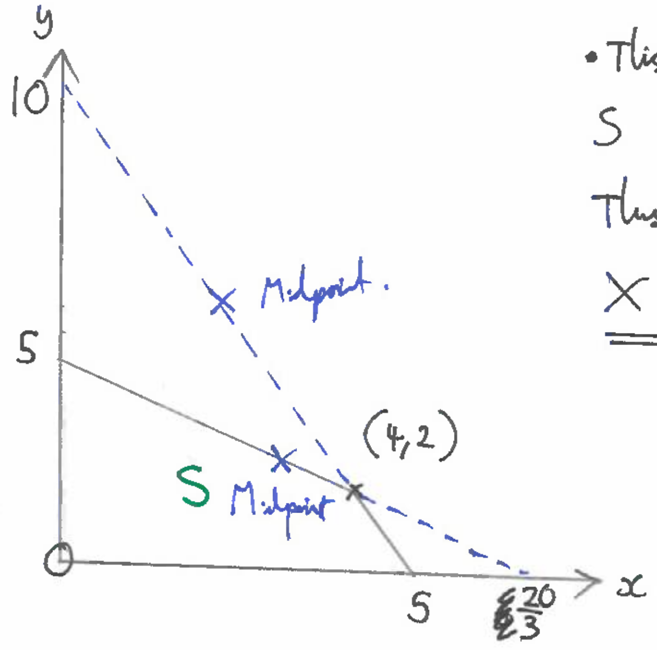
(i).



- clearly from finding the midpoints of the two appropriate tangents, the right line segment forming part of S has midpoint <u>within</u> S.

Thus this gives the Nash bargaining solution where $\underline{\underline{X = 4}}$ and so $\underline{\underline{Y = 4}}$.

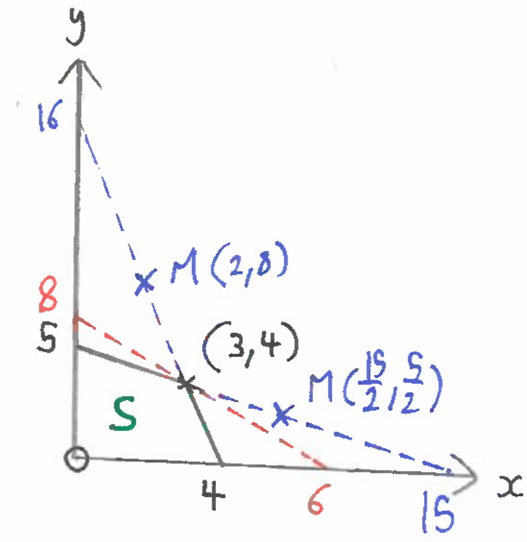
(ii).



- This time the <u>left</u> line segment forming part of S has midpoint <u>within</u> S.

Thus the Nash bargaining solution occurs at $\underline{\underline{X = \frac{10}{3}}}$, $\underline{\underline{Y = \frac{5}{2}}}$

(iii).



• In this case neither line segment defines a tangent whose midpoint belongs to S. This means that the vertex (3, 4) of the bargaining set S is the Nash bargaining solution; this can be verified by considering the red dashed line through $(X, Y) = (3, 4)$ with slope $-\frac{4}{3} = -\frac{Y}{X}$, whose midpoint is indeed (3, 4). This line is a tangent to S.

6. (◇)

a). There are two; one where A proposes: A: M−1, B: 1 and B accepts,
(pure equilibria) one where A proposes: A: M, B: 0 and B accepts,

the second is often deemed a 'weak' equilibrium since B's indifferent between accept and reject, so player A usually takes the first approach as B is willing to accept anything positive. (In practice, with human players, this is not always the approach and A often has to offer much more of the total share for B to accept).

b). (☆)

c). (☆)