Problem Set 5

1). For each of the two-player **cooperative** games labelled a) - d) below:

a).

$A \backslash B$	b_1	b_2
a_1	1, −2	3, 3
a_2	0, 6	3, 1

b).

$A \backslash B$	b_1	b_2
a_1	2, 4	4, 2
a_2	3, 1	1, 3

c).

$A \backslash B$	b_1	b_2
a_1	2, 1	3, 0
a_2	0, 4	2, 5

d).

$A \backslash B$	b_1	b_2
a_1	3, 4	2, 1
a_2	4, 3	1, 1

(i). Find max-min strategies $\hat{\alpha}$ and $\hat{\beta}$ for player A and B using their own payoffs and hence determine the threat point (t_A, t_B).

(ii). Give a sketch of the bargaining set, S, for the game. Label the threat point and indicate the pareto-optimal frontier.

(iii). Find the Nash bargaining solution of the game.

(iv). Show how the players can implement the Nash bargaining solution for S with an appropriate joint strategy.

2). Consider a bargaining set S with threat point $(0, 0)$, let $a > 0$, $b > 0$ and let $S' = \{(ax, by) : (x, y) \in S\}$. Recall that one axiomatic property of the bargaining solution is independence from these scale factors. Show that the Nash product fulfils this property, that is, the Nash bargaining solution from S obtained by maximising the Nash product re-scales to become the solution for S'.

3). Find the Nash bargaining solution in the two-player cooperative game given in normal form below where $a, b \in \mathbb{Z}$ with $a > b > 0$.

		B: b_1	B: b_2
A	a_1	$a-b,\ a+b$	$-b, a$
	a_2	a, b	$0, 0$

4). Two players A and B receive the following payoffs in a cooperative game in which all play agreements are binding:

		B: b_1	B: b_2
A	a_1	$1, 5$	$4, 2$
	a_2	$3, 6$	$1, 1$
	a_3	$4, 5$	$2, 7$

(i). Find the threat point (t_A, t_B) for the cooperative game.

(ii). Sketch the bargaining set for the players and indicate the pareto-optimal frontier.

(iii). Explain why B cannot expect to get more than $\frac{31}{5}$.

(iv). Find the Nash bargaining solution to the game.

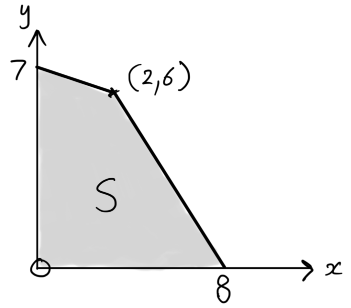
(v). Show how the players can implement the Nash bargaining solution with a joint strategy over the pure strategy pairs in the game.

5). a). Consider a triangle T with vertices $(0,0)$, $(a,0)$ and $(0,b)$ where $a,b>0$. Prove that the rectangle of maximal area contained in T with vertices $(0,0)$, $(X,0)$, (X,Y) and $(0,Y)$ occurs when $X=\frac{a}{2}$ and $Y=\frac{b}{2}$. Furthermore, show that if $S \subseteq T$ and $(X,Y) \in S$, then (X,Y) is also the point in S that maximises the product XY.

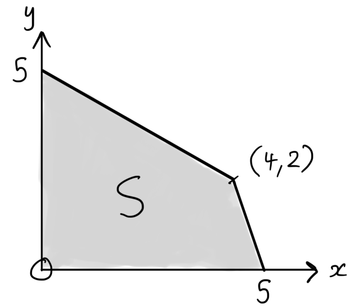
b). Suppose that the pareto-optimal frontier of a bargaining set S with threat point $(0,0)$ is given by $\{(x,f(x)) : 0 \leqslant x \leqslant 1\}$ for a decreasing and continuous function f with $f(0)>0$ and $f(1)=0$. Prove that the bargaining solution (X,Y) is the unique point $(X,f(X))$ where the bargaining set has a tangent line with slope $-\frac{f(X)}{X}$. Show further that if f is differentiable, then this slope is the derivative $f'(X)$ of f at X, i.e $f'(X)=-\frac{f(X)}{X}$.

c). By use of the geometric arguments above; find the Nash bargaining solution in each of the bargaining sets below:

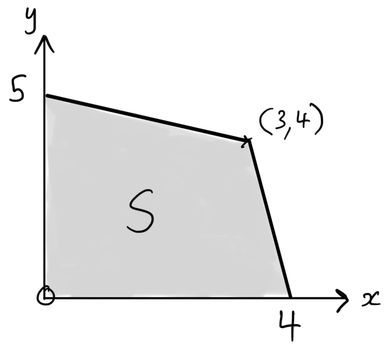
(i).



(ii).



(iii).



6). (◇) **The Ultimatum Game:** Consider the **non-cooperative** game in which a fixed quantity of sweets, M, is to be split between two players, player A and player B. Player A proposes a split of the sweets between the two players; for example with $M=5$, player A may propose the split of 3 sweets for themselves with 2 sweets for player B. This split should be pareto-optimal in the sense that all M sweets should be split, none are to be left unassigned. Player B then decides whether to accept this split, in which case the sweets are assigned and these act as the payoffs to the players, or to reject this split, in which case both players receive nothing!

a). Find all equilibria in this game.

The Ultimatum Game with Reserve Demands: Consider now the same ultimatum game as described above, however, before player A proposes a split, player B first sets a **reserve demand** which is **unknown** to player A. This reserve demand is a number representing the minimum number of sweets that player B will accept in A's proposal. So, for example with $M = 5$, if player B sets their reserve at 3, then they would reject A's proposal of splitting the sweets as 3 for A and 2 for B, meaning both players would receive nothing. However if they had set 1 as their reserve then A's proposal would be accepted and the players A and B would receive 3 and 2 sweets respectively.

b). ($\star$) Perform an analysis of this imperfect information game with reserve demands included (there may be some literature on this that I am currently unaware of).

The Ultimatum Game with Alternating Offers: We can modify the ultimatum game to allow for **alternating offers**, where the players negotiate back and forth with successively converging offers until they can agree on a fair split of the sweets, i.e if B rejects A's initial proposal then they become the proposer and make an offer back to A. Successive proposals from each player then have to improve upon their previous proposal.

c). ($\star$) Under certain constraints (outlined in the reference provided), show that when alternating offers are allowed a 'long' time to converge (are played over the limit of infinitely many rounds of negotiation) that the Nash bargaining solution emerges as the solution in the limit for this game!

If this result interests you see pages 313-330 of the book 'Game Theory Basics' for the details.