Intro to Game Theory: Problem Set 6 Solutions:

1).

a).

(i). The bottom route is *always* worth a cost of 3, regardless of how many users take it. This means that in equilibrium the other two routes must be worth a cost of 2 or 3. This leads to *four* equilibria:

- 3 users top; 2 users middle; 5 users bottom
- 2 users top; 2 users middle; 6 users bottom
- 3 users top; 1 user middle; 6 users bottom
- 2 users top; 1 user middle; 7 users bottom

(ii). Let x users take the top path, y users take the middle path and $10-x-y$ users take the bottom path. Then:

$$\begin{aligned}
\text{Average cost} &= \frac{1}{10}\left(x^2 + y(y+1) + 3(10-x-y)\right) \\
&= \frac{1}{10}\left(x^2 - 3x + y^2 - 2y + 30\right) \\
&= \frac{1}{10}\left(\left(x-\tfrac{3}{2}\right)^2 + (y-1)^2 - \frac{9}{4} - 1 + 30\right) \\
&= \frac{1}{10}\left(x-\tfrac{3}{2}\right)^2 + \frac{1}{10}(y-1)^2 + \frac{107}{40}
\end{aligned}$$

This is clearly minimised when $x = 1$ or 2 and $y = 1$, giving average cost per user of $\frac{1}{10}\left(\frac{1}{2}\right)^2 + \frac{107}{40} = \frac{108}{40} = 2.7$

②

The worst average cost per user in any equilibrium occurs in the first case when all users pay 3, thus the price of anarchy is:

$$POA = \frac{3}{2.7} = 1.11\ldots$$

b). Let x users take the top, y take the middle and $10-x-y$ take bottom.

(i). $Cost_{top} = 2x + y$

$Cost_{mid} = x + y + 2$

$Cost_{bottom} = 10 - x - y + 4 = 14 - x - y$

Now looking at the first two of these shows that we must have $x = 1, 2, 3$ in any equilibrium. Comparing the bottom two means $x+y = 6$, since if $x+y=5$, then $Cost_{bottom} = 9$, but $Cost_{mid} = 7$ and it is advantageous for a user on the bottom to defect to middle reducing their cost to 8, and if $x+y=7$ then $Cost_{bottom} = 7$ but then $Cost_{mid} = 9$ and ~~$Cost_{top}$~~ it is advantageous for a user on middle to defect to the bottom, note that their is also at least one user who would necessarily be on middle (i.e. $y \neq 0$ since $x \neq 7$).

Thus, we check the cases where: $x=1, y=5$, $x=2, y=4$ and $x=y=3$.

We find equilibria when:

- 1 user top; 5 users middle; 4 users bottom
- 2 users top; 4 users middle; 4 users bottom
- ~~3 users top; 3 users middle; 4 users bottom~~

③

(ii). Let x users take the route AB, with z users taking BD above and ~~along the top~~ $x-z$ users taking BD through the middle.
Then $10-x$ users take the bottom route AD.
(I think this would've been easier in part (i) too!)
Then:

$$\text{Average Cost} = \frac{1}{10}\left(x^2+z^2+2(x-z)+(10-x)(14-x)\right)$$
$$= \cdots$$
$$= \frac{1}{5}\left(x-\frac{11}{2}\right)^2+\frac{1}{10}(z-1)^2+\frac{157}{20}$$

This is minimised when $x=5$ or $x=6$ and $z=1$, giving average cost per user of $\frac{1}{5}\left(\frac{1}{2}\right)^2+\frac{157}{20} = \frac{158}{20} = 7.9$

The worst average cost per user in any equilibrium occurs in the second case, when all users pay 8, thus:

$$POA = \frac{8}{7.9} = 1.01\ldots$$

c).

(i). First let's consider the loop route over the top, clearly in any equilibrium this will have either 1 or 2 users. If it has zero it becomes "worth" 3 which performs better than any other route and with three users its cost is 9, worse than using the route of fixed cost 6.
In fact, since there is a route of fixed cost 6 that <u>must</u> be used in any equilibrium (else otherwise it is easy to see the cost of at least one other route exceeds 6), then the costs of the other

routes must be as close as possible to this. This means we can have 2 or 3 users taking $A \to C \to D$ and 2 or 3 users taking $A \to B \to D$, with the remaining users taking $A \to B \to C \to D$ (other than the 1/2 users on the $A \to D$ loop). This gives <u>8</u> equilibria:

~~• 1 user on $A \to D$; 2 users on $A \to C \to D$; 2 users on $A \to B \to D$; 5 users on $A \to B \to C \to D$.~~ We list them in a table:

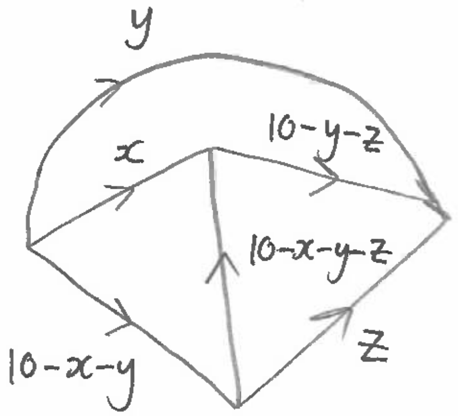
No. users on:

$A \to D$	$A \to C \to D$	$A \to B \to D$	$A \to B \to C \to D$
1	2	2	5
1	2	3	4
1	3	2	4
1	3	3	3
2	2	2	4
2	2	3	3
2	3	2	3
2	3	3	2

(ii). Let y users take the top loop, x users take route $A \to C$, so then $10-x-y$ users take $A \to B$. Let z users take route $B \to D$, so then $10-x-y-z$ users take $B \to C$ and $10-y-z$ users take route $C \to D$.

(5)

※· users :



$z \leqslant 10 - x - y$

$\Rightarrow$ Average Cost $= \frac{1}{10}\left(3y^2 + x^2 + 3(10-x-y) + z^2 + 3(10-y-z)\right)$

$= \cdots$

$= \frac{3}{10}(y-1)^2 + \frac{1}{10}\left(x - \frac{3}{2}\right)^2 + \frac{1}{10}\left(z - \frac{3}{2}\right)^2 + \frac{21}{4}$

This is minimised when $y = 1$, $x = 1$ or 2 and $z = 1$ or 2, giving average cost per user of $\frac{1}{10}\left(\frac{1}{2}\right)^2 + \frac{1}{10}\left(\frac{1}{2}\right)^2 + \frac{21}{4} = \frac{53}{10} = 5.3$.

The worst average cost per user in any equilibrium occurs in the final case in the table where every user pays 6, thus:

$$\text{POA} = \frac{6}{5.3} = 1.13\ldots$$

2).

a). Assume that y users take the bottom route from A to D via B, so that $6-y$ users take the top route. The cost for the bottom route is then $10y + 50 + y = 11y + 50$, and the cost for the top route is $50 + (6-y) + 10(6-y) = 116 - 11y$.

For $y=3$ both costs are equal to 83, which clearly gives an equilibrium because any user would pay more to switch to the other route. Does $y=2$ give an equilibrium? No; the top route then has cost 94, which by switching to the bottom would pay only 83.
for any user on this route

So the equilibrium is with 3 users on each route and is unique.

b). We consider now the right network and allow one user at a time to improve their cost until no further improvement is possible, this will lead to an equilibrium!

First assume there are 3 users on both the top and bottom route. Let there be z users on the new route $A \to B \to C \to D$, currently $z=0$.

When one of the 3 users on the top route, who currently has cost 83, switches to this new route, the flow on edge AB increases from 3 to 4, the flow on edge BC from 0 to 1, and the flow on edge CD remains unchanged. This new route has cost $40+11+30=81 < 83 \Rightarrow$ this user switches to this new route $A \to B \to C \to D$. Now $z=1$ and the loads on each edge are:

Edge	# of users/load	Cost
AB	4	40
AC	2	52
BC	1	11
BD	3	53
CD	3	30

Next, another user who can improve is one of the three users of edge BD who currently pay 53 for use of this edge (plus the cost of 40 for using the edge AB), by using the edges BC and then CD instead. Then BC has two users with cost 12 and CD has four users with cost 40. This new cost is 52 for these two routes, less than 53, hence this user switches to this new path $A \to B \to C \to D$ instead of $A \to B \to D$. Now $Z = 2$ and ~~the~~ we have:

Edge	# of users/load	Cost
AB	4	40
AC	2	52
BC	2	12
BD	2	52
CD	4	40

Now all three paths have total cost 92. This gives us an equilibrium because no user can find a better route. Notice that the equilibrium cost in the left network was 83 and in this network is 92, more costly despite the additional increased capacity.

[This was the example in the original 1968 paper by Braess. The exercise also gives you an idea of an algorithm that could be used to find an equilibrium in a congestion game: Start with some initial spread of the users, and allow one user at a time to improve their costs until no user can improve further].

3). We first consider the situation where every user takes the direct route via one edge to their destination. These are precisely the four edges with cost functions $C(x) = x$, so every user has cost 1. Any other route any user could take would have at least cost 1, so this is both an equilibrium and also the socially optimal flow.

Suppose instead that every user takes their indirect route via two edges. This creates the following loads:

Edge	which users	Cost
AB	2, 4	2
AC	1, 3	2
BA	3	0
BC	2	1
CA	4	0
CB	1	1

This means that users 1 and 2 have cost 3 and users 3 and 4 have cost 2. The average cost per user is $\frac{5}{2} = 2.5$. This is however, bizarrely, an equilibrium! User 1 currently has cost 3, but by switching to the direct route AB would become the third user on that edge and thus pay the same cost. The same holds for user 2. User 3 currently pays 2 and by switching to the direct route BC would become the second user on that edge and also not improve. The same holds for user 4.

These are all equilibria of the game.

4). (☆)(◇)

5). (☆)

6). (☆)

I attach some references that prove/give the intuition behind these results.

As for example congestion games; the game in Q3 has $POA = \frac{5}{2}$ and any Pigou network has $POA = \frac{4}{3}$ using splittable flow model.