Intro to Game Theory: Problem Set 7 Solutions:

1).

a). Consider three piles of sizes $1, m$ and n where $1 \leq m \leq n$.

Observe that: $1,1,m$ is winning as we can move to $1,1$.
$1,m,m$ is winning as we can move to m,m.

(where we use the fact that we know two piles of the same size are losing from the lecture notes).

$1,2,3$ is losing, from lectures, hence $1,2,n$ for $n \geq 4$ is winning, as we can move to $1,2,3$.

$1,3,n$ is winning for $n \geq 3$ by moving to $1,3,2$.

For $1,4,5$, reducing any pile leads to a winning position therefore, so this is a losing position.

The general pattern for these losing positions thus seems to be:
$1, m, m+1$, for <u>even</u> numbers m. This includes the case $m=0$ which we shall take as our base case for induction.

First, let's show that if the positions of form $1,m,n$ with $m \leq n$ are losing when m is even and $n = m+1$, then these are the <u>only</u> losing positions because any other position $1,m,n$ with $m \leq n$ is winning. Indeed, if $m=n$ then a winning move from $1,m,m$ is to m,m, so we can assume $m<n$. If m is even then $n > m+1$ (otherwise we would be in position $1,m,m+1$) and so the winning move is to move to $1,m,m+1$.

If m is odd then the winning move is to $1, m, m-1$.

Second, we show that any move from $1, m, m+1$ with even m is to a winning position, using as our inductive hypothesis that $1, m', m'+1$ for even m' with $m' < m$ is a losing position.

The move to $0, m, m+1$ produces a winning position with counter-move to m, m.

A move to $1, m', m+1$ for $m' < m$ is to a winning position with counter-move to $1, m', m'+1$ if m' is even and to $1, m', m'-1$ if m' is odd.

A move to $1, m, m$ is to a winning position with counter-move to m, m.

A move to $1, m, m'$ with $m' < m$ is also to a winning position with the counter-move to $1, m'-1, m'$ if m' is odd and to $1, m'+1, m'$ if m' is even ($m'+1 < m$ here because m is even).

$\square$

b). Let the three numbers of the pile sizes be $n, n+1$ and $n+2$. If n is even, the reducing the pile of size $n+2$ to 1 creates the position $n, n+1, 1$ which is losing by (a). If n is odd, then $n+1$ is even and $n+2 = (n+1)+1$ so by the same argument a winning move is to reduce the pile of size n to 1. This final argument only works when $n > 1$, hence all positions other than $1, 2, 3$ are winning. $1, 2, 3$ is losing as we know.

c).

	8 4 2 1	winning move 1	winning move 2	winning move 3
8 =	1 0 0 0	→ 0 1 1 0	1 0 0 0	1 0 0 0
11 =	1 0 1 1	1 0 1 1	→ 0 1 0 1	1 0 1 1
13 =	1 1 0 1	1 1 0 1	1 1 0 1	→ 0 0 1 1
Nim Sum =	1 1 1 0	0 0 0 0	0 0 0 0	0 0 0 0

There are 3 possible winning moves:
- reduce 8 to 6
- reduce 11 to 5
- reduce 13 to 3.

2). We prove this by use of top-down induction. Assume that the claim holds for all games simpler than G (call G the impartial game in question). In particular, the claim holds for all options of G. If all the options of G are winning, then G is losing because the other player can force a win no matter which move is made in G. If not all options of G are winning, then one of them, say H, is losing, and by moving to H the player forces a win, hence G is winning.

Taking the game with no options as the base case, the claim holds for all impartial games by top-down induction. □

3). • Transitive: Consider games G, H and J such that $J \leqslant H$ and $H \leqslant G$. This means that there is a sequence of moves from H to J and a sequence of moves from G to H. Playing the second sequence, from G to H, followed by the first sequence, from H to J, means there is a sequence of moves from G to J, hence $J \leqslant G$, as required.

• Reflexive: $G \leqslant G$ since we allow for an empty sequence of moves.

• Antisymmetric: If $J \leqslant G$ and $G \leqslant J$ then if either of these uses a non-empty sequence of moves we violate the ending condition (we can move from G to J and back to G forming an infinitely repeatable series of moves), hence both must contain no moves and so $J = G$, as required.

$\square$

4). • Commutative: $G+H = H+G$, this is clearly true because the order of the options of a game doesn't matter and these games have the same options.

• Associative: $(G+H)+J = G+(H+J)$ holds because both mean in effect that the player decides to move in game G, game H, or in game J, leaving the other two parts unchanged.

• Zero element: $G+0 = G$, indeed the game 0 has no options, so is essentially 'invisible' when added to G: the available options remain the same.

5

5). • Reflexive: $G \equiv G$ is clearly true since $G = G$.

• Symmetric: If $G \equiv H$, then $H \equiv G$ holds from the definition of $\equiv$.

• Transitive: Suppose $G \equiv H$ and $H \equiv K$. Consider another game J. If $G+J$ is losing, then $H+J$ is losing because $G \equiv H$. Then $K+J$ is losing because $H \equiv K$. Conversely, if $K+J$ is losing then $H+J$ is losing and then $G+J$ is losing. Hence $G+J$ losing $\Leftrightarrow$ $K+J$ losing, i.e. by definition, $G \equiv K$ and we have transitivity.

6). ($\diamond$)

7).

a). A player could place a token from their reserve onto a pile/create a new pile, then so could the opponent, then both players could take the same tokens they just played from their reserves back into their reserves, causing an infinite play sequence to occur.

b). The position is losing. The, ordinary, Nim position 3, 2, 1 is losing, so if the player makes a move in the Nim part of the game, the opponent plays an ordinary Nim move to move the position back to a losing position. So can the current player add tokens to form either a new pile or extend old piles to change the game?

No; the second player simply removes the tokens that the first player places onto the game, reverting it back to the original position.

Hence the position is losing in Poker Nim too.

c). Following on from our argument in (b), a player who can win in a position in ordinary Nim, can win in poker Nim. The winning/losing positions are the same, regardless of the numbers of tokens in the players reserves. Reply to the opponent's pile reducing moves just as you would in ordinary Nim, and reverse the effect of any pile-increasing move by using a pile-reducing move to restore the pile to the same size. The ending condition being violated causes no issue here: a player in a winning position wants to end the game with victory, and never has to put back any tokens in doing this; thus the losing player will eventually run out of reserve tokens that can be added and so the game terminates.

8). (◇)

9).

a). Options of: $K_1 : 0$

$K_2 : K_1, 0$

$K_3 : K_2, K_1 + K_1, K_1$

$K_4 : K_3, K_2, K_2 + K_1, K_1 + K_1$

Therefore:

$$K_1 \equiv * \text{mex}(0) = *1$$
$$K_2 \equiv * \text{mex}(1, 0) = *2$$
$$K_3 \equiv * \text{mex}(2, 1 \oplus 1, 1) = * \text{mex}(2, 0, 1) = *3$$
$$K_4 \equiv * \text{mex}(3, 2, 2 \oplus 1, 1 \oplus 1) = * \text{mex}(3, 2, 3, 0) = *1$$

b). Consider K_n, where n is odd. Player 1 can knock out pin $\frac{n+1}{2}$ on their first move, leaving player 2 with the position

$$K_{\frac{n-1}{2}} + K_{\frac{n-1}{2}} \equiv 0, \text{ by the Copycat principle.}$$

Now any option of player 2 occurs in one of these games $K_{\frac{n-1}{2}}$, so player 1 can make the corresponding move in the other of these games to leave player 2 in a losing position once again.
This can be formalised with top-down induction + the base case.

Consider K_n, where n is even. Player 1 can instead knock out pins $\frac{n}{2}$ and $\frac{n}{2}+1$ on their first move, leaving player 2 with position

$$K_{\frac{n}{2}-1} + K_{\frac{n}{2}-1} \equiv 0, \text{ by the copycat principle.}$$

A similar argument to the above now follows.
Hence player 1 always has a winning strategy in Kayles.

c). ($\Diamond$) In K_n, knocking out a single pin, p, leaves the game: $K_{p-1} + K_{n-p}$, and knocking out pins $p, p+1$, leaves the game: $K_{p-1} + K_{n-p-1}$. The options of K_n are these game sums for all possible p values (where p need only be considered up to the middle pin due to symmetry).
Using this recursively with the mex rule, only the Nim values of these positions are important.

10).

a). For $2\times n$ Chomp, player A can always force a win. The winning move is to remove the bottom-right dot $(2,n)$, leaving a pattern like: (e.g. shown for $n=5$)

Afterwards, player A can always re-create this pattern by removing the dot that is diagonally adjacent to the dot that player B removes. i.e. any move of player B of form $(1,i)$, $i>1$ can be countered by $(2,i-1)$, and any move $(2,i)$ countered by $(1,i+1)$. Then player A always has a move left and wins.

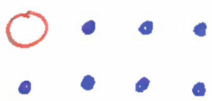
For $m\times m$ square games where $m\geqslant 2$, the winning move is $(2,2)$. Then player A can respond to a move of type $(i,1)$ by removing $(1,i)$ and vice-versa until player B is forced to take $(1,1)$ and loses.

b). We remove the 'poisoned' dot at $(1,1)$. Then the last player loses by not being able to move any more, exactly when before they would have had to take the 'poisoned' dot.

c). Consider removing the bottom-right dot (m,n). If this is a winning move, we are done. If not, there is a counter-move (i,j) by player B that would create a losing situation for player A. But in that case, player A could have just made the move (i,j) to begin with, creating the same losing situation for player B. Thus the game is <u>winning for player A</u>, even though we don't know what is the winning move!

When $n=m$ however, part (a) gives a known winning solution.

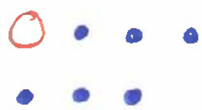
d). We first represent Chomp in the normal play convention using part (b). We remove the top-left dot, indicated by a red-empty circle in the list of ~~options~~ positions below, we label each ~~option~~ position in the Chomp game with a letter, where A represents the initial position:

A: 

B: 

C: 

D: 

E: 

F: 

G: 

H: 

I: 

J: 

K: 

L: 

M: 

~~Positions~~ The Nim values of the simpler games are easy to compute, for example, B is $*3$ because it acts like a single Nim pile of size 3. The game C acts like two Nim piles of sizes 1 and 3, so has Nim value 2 because $C \equiv *1 + *3 \equiv *2$.

Similarly, we find:

$$H \equiv *1$$
$$I \equiv *1$$
$$J \equiv *2$$
$$K \equiv *1 + *2 \equiv *3$$
$$M \equiv *1 + *1 \equiv 0$$

Now we use the mex rule for the more complicated positions:

- The options of G are H, I and M, so:

$$G \equiv * \text{mex}(1,1,0) \equiv *2$$

- The options of L are G, H, J and K, so:

$$L \equiv * \text{mex}(2,1,2,3) \equiv 0$$

- The options of F are G, H, J, K and L, so:

$$F \equiv * \text{mex}(2,1,2,3,0) \equiv *4$$

- The options of D are B, C, G, H and L, so:

$$D \equiv * \text{mex}(3,2,2,1,0) \equiv *4$$

- The options of E are B, C, D, F, G and H, so:

$$E \equiv * \text{mex}(3,2,4,4,2,1) \equiv 0$$

- Finally, the options of A are B, C, D, E, F, G and H, so:

$$A \equiv * \text{mex}(3,2,4,0,4,2,1) \equiv *5$$

Now we can find all the possible winning moves. We are currently in

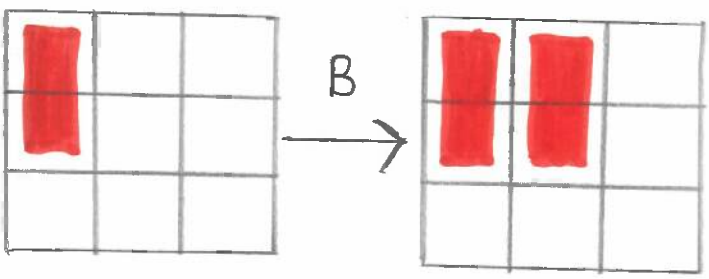
$$A + *4 \equiv *5 + *4,$$

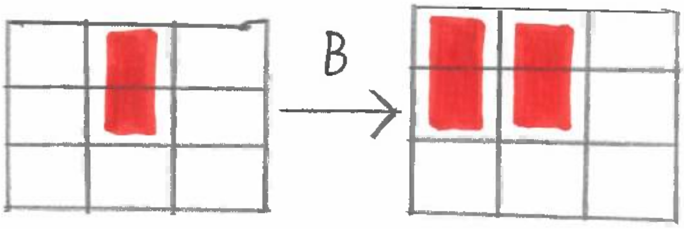
but the Nim pile of size 4 cannot be increased to size 5, hence the possible winning moves are to move to an option of A that has Nim value 4 (giving $*4 + *4 \equiv 0$, a losing position). There are therefore two possible winning moves, moving the chomp part of the game sum to positions D or F.

11).

a).

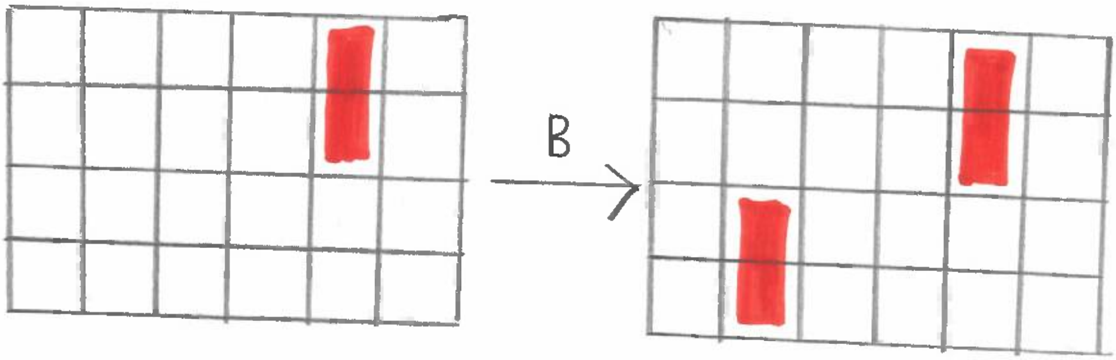
(i). In 3×3 Cram there are two possible opening moves for player A up to symmetry; namely placing the domino such that it occupies a corner square or such that it occupies the centre square. In either case, player B can respond by placing their domino alongside the first domino, such that a 2×2 square in one corner is occupied, leaving an L-shape of five squares remaining:





Regardless of what player A now does in this L-shaped configuration, player B will always be able to place the last domino and player A loses. Hence, 3×3 Cram is a win for player B.

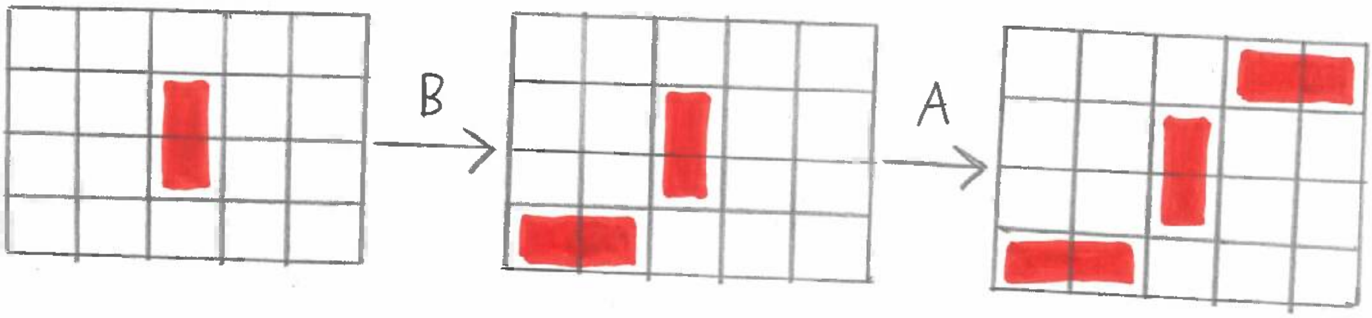
(ii). When both m and n are even, player B will win in $m \times n$ Cram by playing 'copycat' using the central symmetry of the board; i.e. wherever player A places their domino, player B responds by placing their domino on the squares obtained by performing a $180°$ rotation of player A's domino about the centre-point of the board, e.g. for a 4×6 board:



Player B's squares are <u>always</u> available because player A's squares were available before player A made their move.

(13)

(iii). In $m \times n$ Cram where one is odd and the other even, player A has a winning strategy by placing their first domino on the central pair of squares (in the middle row/column, whichever was odd), and then preceeding to play 'copycat' exactly as in part (ii). e.g. on a 4×5 board play could start:



b).

(i). Putting the domino anywhere on the board produces two independent boards of size $1 \times k$ and $1 \times (n-k-2)$. The resulting position is a sum of two games because the player can only move in one of them, which is equivalent to the sum of Nim piles $*D_k + *D_{n-k-2}$.
By symmetry, only $k \leq \frac{n}{2} - 1$ need to be considered.

Hence:

$$D_n = \text{mex}\left(\left\{ D_k \oplus D_{n-k-2} : 0 \leq k \leq \frac{n}{2} - 1 \right\}\right).$$

(ii). Using the result from (i), with the cases: $D_0 = D_1 = 0$ as losing games.

~~Start the~~ $D_2 = \text{mex}(D_0 \oplus D_0) = \text{mex}(0) = 1,$

$D_3 = \text{mex}(D_0 \oplus D_0) = \text{mex}(0) = 1,$

$$D_4 = \text{mex}(D_0 \oplus D_2, D_1 \oplus D_1) = \text{mex}(1,0) = 2,$$

$$D_5 = \text{mex}(D_0 \oplus D_3, D_1 \oplus D_2) = \text{mex}(1,1) = 0,$$

$$D_6 = \text{mex}(D_0 \oplus D_4, D_1 \oplus D_3, D_2 \oplus D_2) = \text{mex}(2,1,0) = 3,$$

$$D_7 = \text{mex}(D_0 \oplus D_5, D_1 \oplus D_4, D_2 \oplus D_3) = \text{mex}(0,2,0) = 1,$$

$$D_8 = \text{mex}(D_0 \oplus D_6, D_1 \oplus D_5, D_2 \oplus D_4, D_3 \oplus D_3)$$
$$= \text{mex}(3,0,3,0) = 1,$$

$$D_9 = \text{mex}(D_0 \oplus D_7, D_1 \oplus D_6, D_2 \oplus D_5, D_3 \oplus D_4)$$
$$= \text{mex}(1,3,1,3) = 0,$$

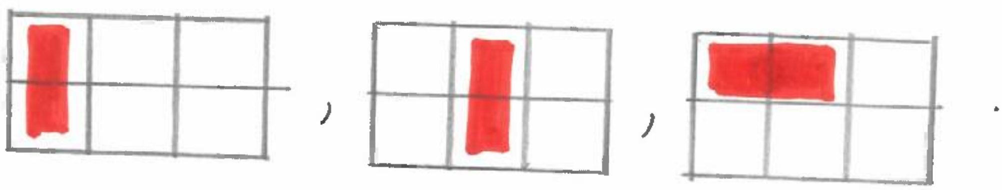
$$D_{10} = \text{mex}(D_0 \oplus D_8, D_1 \oplus D_7, D_2 \oplus D_6, D_3 \oplus D_5, D_4 \oplus D_4)$$
$$= \text{mex}(1,1,2,1,0) = 3.$$

others beyond $n=10$

Cram on a $1 \times n$ board is losing for $n = 0, 1, 5, 9, \ldots$

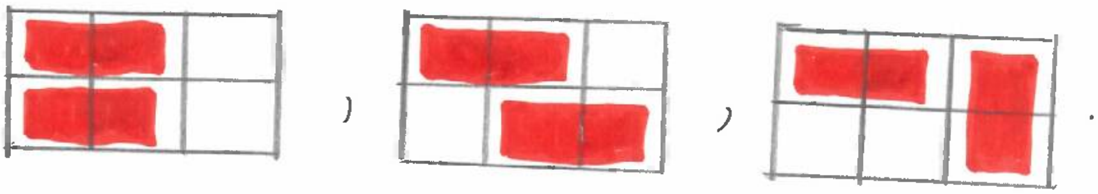
(iii). (◇) The sequence of D_n values eventually begins to repeat with period 34. The highest occuring Nim value is 9, interestingly no 6 ever appears.

c). Cram on a 2×3 board has, up to symmetry, the options:



The first two of these are clearly losing because exactly two more dominos will be placed subsequently.

The third of these has the options:



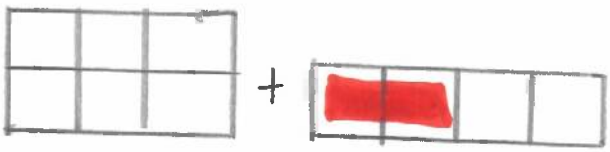
These have Nim values 1, 0 and 1 respectively, so the Nim value of the third option is mex$(1,0,1) = 2$.

The Nim value of the 2×3 board is therefore mex$(0,0,2) = 1$.

d). The Nim value of 2×3 Cram is 1 and of 1×4 Cram is 2 (from (c) and (b) respectively).

So the game sum is equivalent to $*1 + *2$ and is therefore winning.

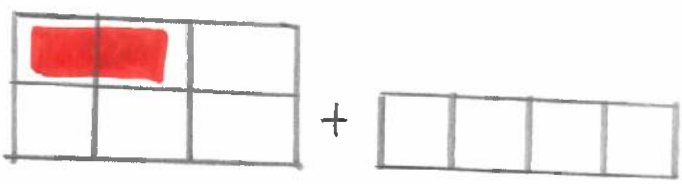
There are two winning moves: the first is to reduce the 'second Nim pile' to size 1, which constitutes placing a domino at one end of the 1×4 board as shown:



(or count this as 2 winning moves without symmetry)

$1 \oplus 1 = 0$

The second winning move is to increase the 'first Nim pile' to size 2 because this was one of the options of the 2×3 board. This corresponds to playing the domino in a corner:



(or count this as 4 winning moves without symmetry).

$2 \oplus 2 = 0$

12).

a). We will show that $G+H$ is losing by showing that every option of $G+H$ is winning.

Consider an option $G'+H$, where G' is an option of G. By the initial assumption, there is an option H' of H with $G' \equiv H'$, that is, $G'+H' \equiv 0$ by lemma 7.62 from the lecture notes.
So $G'+H'$ is losing, so moving there defines a winning move in the game $G'+H$.
By the same argument, any option $G+H'$ of $G+H$ is also winning, so as claimed, <u>every</u> option of $G+H$ is winning.
Therefore $G+H \equiv 0$ is losing and $G \equiv H$ by lemma 7.62. $\square$

b). ~~But~~ p and q can be written as sums of distinct powers of 2 smaller than 2^d. By theorem 7.68, r is also a sum of ~~k~~ powers of 2 less than 2^d where the powers that appear in both p and q cancel out in the game sum because $*(2^k) + *(2^k) \equiv 0$. The resulting sum r is therefore also less than 2^d. $\square$

c). (☆) See the text "Game Theory Basics" by Von Stengel pages 21–22 "proof of theorem 1.15" for a proof of this result
(some notation is slightly different but should be understandable).

13). (☆)(◇) Ask the lecturer for texts/ideas if interested.