Problem Set 7

1). This question is about the game of **Nim**.

a). For all positions with **three** piles, where one of the piles has only **one** token, describe exactly the losing positions. Justify your answer, for example with a proof by induction, or by theorems on Nim, but try to do this **without** use of the binary representation seen in lectures.

b). Consider a Nim game with three piles whose sizes are consecutive integers, for example $3, 4, 5$. Show that the **only** case where this is a losing position is $1, 2, 3$ and that otherwise this is always a winning position.

c). Determine all initial winning moves for Nim with three piles of sizes 8, 11 and 13, using the binary representation seen in lectures.

2). Prove proposition 7.43 from the lecture notes: In an impartial game, a game position is losing if and only if all its options are winning positions. A game position is winning if and only if at least one of its options is a losing position; moving to that position is a winning move.

3). Prove proposition 7.49 from the lecture notes: the binary relation $\leqslant$ on a set S of games forms a partial order.

4). Prove proposition 7.53 from the lecture notes: the game sum, $+$, is commutative, associative and the zero game is a zero element for $+$.

5). Prove lemma 7.55 from the lecture notes: $\equiv$ is an equivalence relation between games.

6). ($\diamond$) **Misére Nim:** Misére Nim is played just like Nim but with the misére play convention, so the last player to move loses. A losing position is therefore a single Nim pile with a single token in it (which the player has to take). Another losing position is three piles with a single token each.

a). Determine the winning and losing positions in misére Nim with one or two piles, and the winning moves from a winning position.

b). Are three piles of sizes 3, 2 and 1 winning or losing in misére Nim?

c). Describe the losing positions in misére Nim for **any** number of piles.

7). **Poker Nim:** Poker Nim is a variation of Nim. This game is played like Nim, but at the beginning of the game each player is given some extra ‘reserve’ tokens (of white or colourless colour - they will act like ‘jokers’ or ‘wildcards’). In a move, a player can choose, as in ordinary Nim, a pile and remove some tokens which they can then **add** to their reserve (where they then become white/colourless).

A second new kind of move available is to **add** some (at least one, possibly all) of the player's reserve tokens to some pile, or even to create an entirely new pile with these tokens. We use the normal play convention, so the first player who cannot move loses the game.

a). Strictly speaking, the ending condition is violated in poker Nim. Discuss why this is the case.

b). Consider the position $3, 2, 1$ and where both players have 3 tokens in their reserve. Identify if this position is winning/losing. Justify your answer.

c). Classify all winning/losing positions in poker Nim. Describe why the violation of the ending condition is not a problem in your analysis.

8). ($\diamond$) Consider a set S of impartial games consisting of some game and all games simpler than this game. Prove that $(S, +)$ (the set S with the binary operation of $+$ between games) forms an **abelian group** when we use $\equiv$ rather than $=$ for determining game sums.

9). **Kayles Game:** This is an impartial game played as follows: given a row of n bowling pins (numbered $1, 2, \cdots, n$), a move knocks out one or two **consecutive** pins. For example, the figure shows a move where a game of 5 pins has pins 2 and 3 knocked out. The game is played with the normal play convention, so knocking out the final pin results in winning.

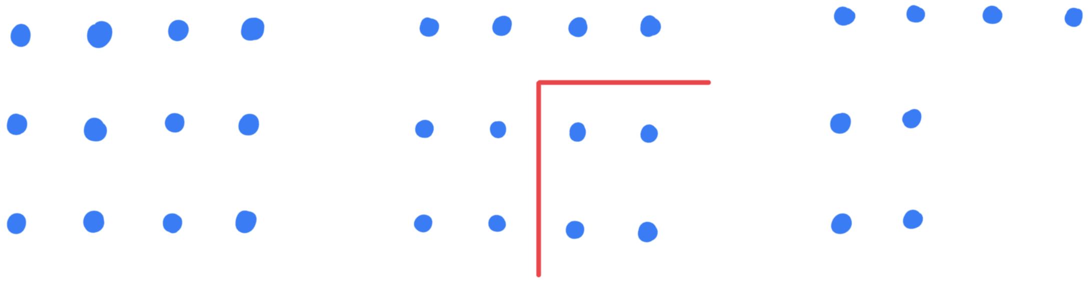
We denote Kayles game with n pins by K_n.

a). Find the Nim values of K_1, K_2, K_3 and K_4.

b). Prove that one of the players always has a guaranteed winning strategy in Kayles game.

c). ($\diamond$) Determine the Nim value of K_n for **any** value of n (Note: use computational help for this via recursion and you should find that after a certain point the values become periodic).

10). **Chomp:** Consider a rectangular array of $m \times n$ dots in m rows and n columns, for example the 3×4 array shown on the right in the figure below.
A dot in row i, column j is named (i, j). This is an impartial game where a move consists of selecting a dot (i, j) and then removing it and **all other dots to the right and below it**, which means removing all dots (i', j') such that $i' \geqslant i$ and $j' \geqslant j$, as shown in the centre and right of the figure for the choice $(i, j) = (2, 3)$.



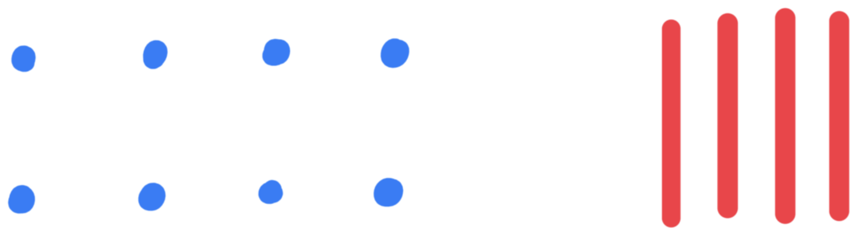
Players alternate choosing dots in this game but the usual convention in this game is to take **misére** play, so that the player who removes the final dot, the top-left hand dot (which you can think of as being poisoned) **loses** the game.

a). Assuming optimal play, determine the winning player and a winning move for Chomp of size 2×2, size 2×3, size $2 \times n$, and size $m \times m$, where $m \geqslant 3$. Justify your answers.

b). Suppose we want to play the '**same**' game but insist that the **normal** play convention applies, where the last player to move wins. How should the array of dots be changed to achieve this? (Note that normal play with the array as given is extremely boring, the first player simply takes the top-left dot $(1, 1)$.)

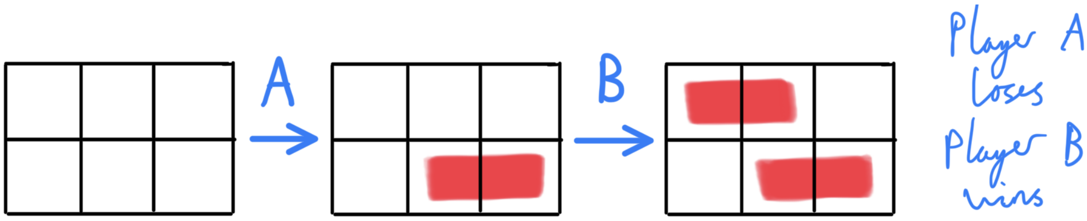
c). Show that when Chomp is played for a game of any size $m \times n$, the first player can **always** win (unless $m = n = 1$). [Hint: you only need to show a winning move **exists**, you need not describe what it is.]

d). Consider now the game Chomp of size 2×4 in a game sum with a Nim pile of size 4, as shown.



What are the winning moves of the starting player, if any?

11). **Cram:** The impartial game Cram is played on a board of $m \times n$ squares, where players alternately place a domino on the board which covers two adjacent squares that are free (not yet occupied by a domino), vertically or horizontally. The first player who cannot place a domino anymore loses. The figure below shows an example play of the game on a 2×3 board.



a). (i). Who will win in 3×3 Cram?

(ii). Who will win in $m \times n$ Cram when both m and n are even?

(iii). Who will win in $m \times n$ Cram when one of m and n is odd and the other even?

b). Now consider Cram played on a $1 \times n$ board, where $n \geqslant 2$. Let D_n be the Nim value of this game.

(i). How is D_n computed from smaller values D_k for $k < n$?

(ii). Find the values of D_n up to $n = 10$. For which values of n is Cram on a $1 \times n$ board a losing game?

(iii). ($\diamond$) Continue recursively beyond $n = 10$ with computational help.

c). Find the Nim value of 2×3 Cram.

d). Find all winning moves, if any, for the game sum of a 2×3 Cram game and a 1×4 Cram game.

12). a). Consider two impartial games G and H such that for every option of G there is an equivalent option of H and vice versa. Prove then that $G \equiv H$.

b). Suppose that theorem 7.68 holds from the lecture notes for any n with $n < 2^d$. Let $0 \leqslant p, q < 2^d$. Prove then that $*p + *q \equiv *r$, where $0 \leqslant r < 2^d$, i.e that $r = p \oplus q < 2^d$.

c). ($\star$) With the help of parts (a) and (b), prove theorem 7.68 from the lecture notes.

13). ($\star$)($\diamond$) Investigate some of the theory behind Partizan games or investigate a particular Partizan game you're interested in.