Optimisation - MATH70005

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Contents

1 Mathematical Preliminaries

1.1 Vector Spaces

Definition 1.1 (Vector Space)**.** A *vector space* over a field $\mathbb{F}$ is a set V equipped with two operations: vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ such that the following properties hold:

1. **Closure under Addition:** For all $u, v \in V$, $u + v \in V$.
2. **Associativity of Addition:** For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
3. **Commutativity of Addition:** For all $u, v \in V$, $u + v = v + u$.
4. **Additive Identity:** There exists a vector $0 \in V$ such that for all $v \in V$, $v + 0 = v$.
5. **Additive Inverse:** For all $v \in V$, there exists a vector $-v \in V$ such that $v + (-v) = 0$.
6. **Closure under Scalar Multiplication:** For all $\lambda \in \mathbb{F}$ and $v \in V$, $\lambda v \in V$.
7. **Distributivity of Scalar Multiplication over Vector Addition:** For all $\lambda \in \mathbb{F}$ and $u, v \in V$, $\lambda(u + v) = \lambda u + \lambda v$.
8. **Distributivity of Scalar Multiplication over Field Addition:** For all $\lambda, \mu \in \mathbb{F}$ and $v \in V$, $(\lambda + \mu)v = \lambda v + \mu v$.
9. **Compatibility of Scalar Multiplication with Field Multiplication:** For all $\lambda, \mu \in \mathbb{F}$ and $v \in V$, $(\lambda\mu)v = \lambda(\mu v)$.
10. **Identity Element of Scalar Multiplication:** For all $v \in V$, $1v = v$.

Definition 1.2 (Inner Product)**.** An *inner product* on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that for all $u, v, w \in V$ and $\lambda \in \mathbb{F}$, the following properties hold, considering only real vector spaces:

1. **Symmetry:** $\langle u, v \rangle = \langle v, u \rangle$.
2. **Additivity:** $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.
3. **Homogeneity:** $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle, \forall \lambda \in \mathbb{R}$.
4. **Positive Definiteness:** $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Definition 1.3 (Norm)**.** The *norm* of a vector $v \in V$ is defined as $\|v\| = \sqrt{\langle v, v \rangle}$. The norm satisfies the following properties:

1. $\|v\| \geq 0$ with equality if and only if $v = 0$.

2. $\|\lambda v\| = |\lambda|\|v\|$.

3. $\|u+v\| \leq \|u\| + \|v\|$.

Theorem 1.4 (Cauchy-Schwarz Inequality)**.** *For all $u, v \in V$, $|\langle u, v\rangle| \leq \|u\|\|v\|$. Equality holds if and only if u and v are linearly dependent.*

Definition 1.5 (Matrix Norms)**.** A norm $|\cdot|$ on $\mathbb{R}^{m\times n}$ is a function $|\cdot| : \mathbb{R}^{m\times n} \to \mathbb{R}$ such that for all $A, B \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$, the following properties hold:

1. **Non-negativity:** $|A| \geq 0$ with equality if and only if $A = 0$.

2. **Homogeneity:** $|\lambda A| = |\lambda|\,|A|$.

3. **Triangle Inequality:** $|A+B| \leq |A| + |B|$.

Definition 1.6 (Induced Norms)**.** Given matrix $A \in \mathbb{R}^{m\times n}$ and two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, the *induced norm* (a, b) norm) of A is defined as:

$$\|A\|_{a,b} = \max_{\mathbf{x}}\{\|\mathbf{Ax}\|_b : \|\mathbf{x}\|_a \leq 1\}$$

Example 1.7 (Examples of Norms)**.** 1. **Spectral Norm:** ((2,2)-norm): $\|A\|_2 = \sigma_{\max}(A)$.

2. ℓ_1 **norm:** $\|A\|_1 = \max_j \sum_i |a_{ij}|$.

3. ℓ_∞ **norm:** $\|A\|_\infty = \max_i \sum_j |a_{ij}|$.

4. **Frobenius Norm:** $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$.

1.2 Eigenvalues and Eigenvectors

Definition 1.8 (Eigenvalues and Eigenvectors)**.** Let $A \in \mathbb{R}^{n\times n}$. A scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there exists a non-zero vector $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an *eigenvector corresponding to the eigenvalue* λ.

Theorem 1.9 (Spectral Factorization Theorem)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then, there exists an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.*

The columns of Q are the eigenvectors of A and the diagonal elements of Λ are the corresponding eigenvalues of A.

Corollary 1.10 (Trace and Determinant)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, the trace and determinant of A are given by:*

$$tr(A) = \sum_{i=1}^{n} \lambda_i \quad and \quad det(A) = \prod_{i=1}^{n} \lambda_i$$

Corollary 1.11 (Rayleigh Quotient Bound)**.** *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Then, for any vector $\mathbf{x} \in \mathbb{R}^n$, the Rayleigh quotient satisfies:*

$$\lambda_1 \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \leq \lambda_n$$

1.3 Basic Topological Concepts

Definition 1.12 (Balls)**.** An *open ball* of radius $\epsilon > 0$ centered at a point $\mathbf{x} \in \mathbb{R}^n$ is defined as:

$$B(\mathbf{x}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| < \epsilon\}$$

A *closed ball* of radius $\epsilon > 0$ centered at a point $\mathbf{x} \in \mathbb{R}^n$ is defined as:

$$\bar{B}(\mathbf{x}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \leq \epsilon\}$$

Definition 1.13 (Interior Point)**.** A point $\mathbf{x} \in \mathbb{R}^n$ is an *interior point* of a set $S \subseteq \mathbb{R}^n$ if there exists an open ball centered at $\mathbf{x}$ that is contained in S. The set of all interior points of S is denoted by $\text{int}(S)$.

$$\text{int}(S) = \{\mathbf{x} \in S : \text{there exists an open ball } B(\mathbf{x}, \epsilon) \subseteq S\}$$

Definition 1.14 (Closed Set)**.** A set $S \subseteq \mathbb{R}^n$ is *closed* if it contains all its limit points. A limit point of a set S is a point $\mathbf{x} \in \mathbb{R}^n$ such that every open ball centered at $\mathbf{x}$ contains a point in S.

We have that a set U is closed $\iff U^c$ is open.

Definition 1.15 (Boundary Points)**.** A point $\mathbf{x} \in \mathbb{R}^n$ is a *boundary point* of a set $S \subseteq \mathbb{R}^n$ if every open ball centered at $\mathbf{x}$ contains points in S and points not in S. The set of all boundary points of S is denoted by ∂S.

$$\partial S = \{\mathbf{x} \in \mathbb{R}^n : \text{every open ball centered at } \mathbf{x} \text{ contains points in } S \text{ and points not in } S\}$$

Definition 1.16 (Closure)**.** The closure of a set $S \subseteq \mathbb{R}^n$ is the union of S and its boundary points. Equivalently the closure of a set S is the smallest closed set containing S.

$$\bar{S} = S \cup \partial S, \quad \bar{S} = \bigcap \{U \subseteq \mathbb{R}^n : U \text{ is closed and } S \subseteq U\}$$

1.4 Directional Derivatives and Gradients

Definition 1.17 (Directional Derivative)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $\mathbf{x} \in \mathbb{R}^n$. The *directional derivative* of f at $\mathbf{x}$ in the direction of a vector $\mathbf{d} \in \mathbb{R}^n$ is defined as:

$$\nabla_{\mathbf{d}} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x})}{t}$$

Definition 1.18 (Continuous Differentiability)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuously differentiable* at a point $\mathbf{x} \in \mathbb{R}^n$ if all the partial derivatives of f exist and are continuous in a neighbourhood of $\mathbf{x}$.

Proposition 1.19. *Let $f : U \to \mathbb{R}$ defined on open set $U \subseteq \mathbb{R}^n$. Suppose that f is continuously differentiable over U. Then*

$$\lim_{d\to 0} \frac{f(\mathbf{x}+\mathbf{d}) - f(\mathbf{x}) - \nabla f(\mathbf{x})^T \mathbf{d}}{\|\mathbf{d}\|} = 0, \quad \forall \mathbf{x} \in U$$

Or equiv:

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + o(\|\mathbf{y}-\mathbf{x}\|)$$

where $o(\|\mathbf{y}-\mathbf{x}\|)$ is the Landau notation for a function that vanishes faster than a linear function. i.e $\frac{o(t)}{t} \to 0$ as $t \to 0$

Definition 1.20 (Twice differentiablility and Hessian)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The *second order partial derivatives* of f at a point $\mathbf{x} \in \mathbb{R}^n$ are defined as:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

The *Hessian matrix* of f at $\mathbf{x}$ is defined as:

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Theorem 1.21 (Linear Approximation Theorem)**.** *Let $f : U \to \mathbb{R}$ be a function defined on $U \subseteq \mathbb{R}^n$, that is twice continuously differentiable over U . Let $\mathbf{x} \in U$ and $\epsilon > 0$ satisfying $B(x, \epsilon) \subseteq U$. Then for any $y \in B(x, \epsilon)$, there exists $\xi \in [\mathbf{x}, \mathbf{y}]$ such that:*

$$f(\mathbf{y}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^T \nabla^2 f(\xi)(\mathbf{y}-\mathbf{x})$$

Theorem 1.22 (Quadratic Approximation Theorem)**.** *Let $f : U \to \mathbb{R}$, defined on open set $U \subseteq \mathbb{R}^n$. Suppose f is twice continuously differentiable over U. Let $\mathbf{x} \in U$ and $r > 0$ satisfying $B(\mathbf{x}, r) \subseteq U$. Then for any $\mathbf{y} \in B(\mathbf{x}, r)$:*

$$f(y) = f(x) + \nabla f(x)^T(\mathbf{y}-\mathbf{x}) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^T \nabla^2 f(\mathbf{x})(\mathbf{y}-\mathbf{x}) + o(\|\mathbf{y}-\mathbf{x}\|^2)$$

Proposition 1.23 (Gradient of Linear Function)**.** *Given function $f(\mathbf{w}) = \mathbf{a}^T\mathbf{w}$. Then*

$$\nabla f(\mathbf{w}) = \mathbf{a}$$

Proposition 1.24 (Gradient of Quadratic Function)**.** *Given function* $f(\mathbf{w}) = \mathbf{w}^T A\mathbf{w}$*. Then*

$$\nabla f(\mathbf{w}) = (A + A^T)\mathbf{w}$$

Note that if A *is symmetric, then* $\nabla f(\mathbf{w}) = 2A\mathbf{w}$
More generally for a function $f(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T A\mathbf{w} + \mathbf{b}^T w + \gamma$*, the gradient is given by:*

$$\nabla f(\mathbf{w}) = \frac{1}{2}(A^T + A)\mathbf{w} + \mathbf{b}$$

And again if A *is symmetric, then* $\nabla f(\mathbf{w}) = A\mathbf{w} + \mathbf{b}$

Proposition 1.25 (Hessian of Quadratic Function)**.** *For function of the form* $f(\mathbf{w}) = \mathbf{w}^T A\mathbf{w}$*. The Hessian is given by:*

$$\nabla^2 f(\mathbf{w}) = A + A^T$$

Which for A *symmetric simplifies to* $\nabla^2 f(\mathbf{w}) = 2A$

2 Unconstrained Optimisation

2.1 Global Minimum and Maximum

Definition 2.1 (Global Minimum and Maximum)**.** Let $f : S \to \mathbb{R}$, defined on set $S \subseteq \mathbb{R}^n$. Then

1. $\mathbf{x}^* \in S$ a *global minimum* of f over S if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.
2. $\mathbf{x}^* \in S$ a *strict global minimum* of f over S if $f(\mathbf{x}^*) < f(\mathbf{x})$ for all $\mathbf{x} \in S, \mathbf{x} \neq \mathbf{x}^*$.
3. $\mathbf{x}^* \in S$ a *global maximum* of f over S if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in S$.
4. $\mathbf{x}^* \in S$ a *strict global maximum* of f over S if $f(\mathbf{x}^*) > f(\mathbf{x})$ for all $\mathbf{x} \in S, \mathbf{x} \neq \mathbf{x}^*$.

We denote by **global optimum** the global minimum or maximum.

- maximal value of f over S
$$\sup\{f(\mathbf{x}) : \mathbf{x} \in S\}$$
- minimal value of f over S
$$\inf\{f(\mathbf{x}) : \mathbf{x} \in S\}$$

2.2 Local Minima and Maxima

Definition 2.2 (Local Minimum and Maximum)**.** Let $f : S \to \mathbb{R}$ be defined on a set $S \subseteq \mathbb{R}^n$. Then:

1. $\mathbf{x}^* \in S$ a *local minimum* of f over S if there exists $r > 0$ for which $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for any $\mathbf{x} \in S \cap B(\mathbf{x}^*, r)$
2. $\mathbf{x}^* \in S$ a *strict local minimum* of f over S if there exists $r > 0$ for which $f(\mathbf{x}^*) < f(\mathbf{x})$ for any $\mathbf{x} \neq \mathbf{x}^* \in S \cap B(\mathbf{x}^*, r)$
3. $\mathbf{x}^* \in S$ a *local maximum* of f over S if there exists $r > 0$ for which $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for any $\mathbf{x} \in S \cap B(\mathbf{x}^*, r)$
4. $\mathbf{x}^* \in S$ a *strict local maximum* of f over S if there exists $r > 0$ for which $f(\mathbf{x}^*) > f(\mathbf{x})$ for any $\mathbf{x} \neq \mathbf{x}^* \in S \cap B(\mathbf{x}^*, r)$

Theorem 2.3 (Fermat's Theorem: First Order Optimality Conditions)**.** *Let $f : U \to \mathbb{R}$ a function defined on set $U \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}^* int(U)$ a local optimum point and all partial derivatives of f exist at $\mathbf{x}^*$. Then:*

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

Definition 2.4 (Stationary Points)**.** Let $f : U \to \mathbb{R}$ a function defined on set $U \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}^* \in \text{int}(U)$ and all partial derivatives of f exist at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a *stationary point* of f if:

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

2.3 Second Order Optimality Conditions

Theorem 2.5 (Necessary Second Order Optimality Conditions)**.** *Let $f : U \to \mathbb{R}$ a function defined on open set $U \subseteq \mathbb{R}^n$. Suppose f is twice continuously differentiable over U and $\mathbf{x}^*$ a stationary point. Then:*

1. *if $\mathbf{x}^*$ a local minimum point, then $\nabla^2 f(\mathbf{x}^*) \succeq 0$*
2. *if $\mathbf{x}^*$ a local maximum point, then $\nabla^2 f(\mathbf{x}^*) \preceq 0$*

Theorem 2.6 (Sufficient Second Order Optimality Conditions)**.** *Let $f : U \to \mathbb{R}$ a function defined on open set $U \subseteq \mathbb{R}^n$. Suppose f is twice continuously differentiable over U and $\mathbf{x}^*$ a stationary point. Then:*

1. *if $\nabla f(\mathbf{x}^*) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}^*) \succ 0$, then $\mathbf{x}^*$ a strict local minimum point*
2. *if $\nabla f(\mathbf{x}^*) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}^*) \prec 0$, then $\mathbf{x}^*$ a strict local maximum point*

2.4 Saddle Points

Definition 2.7 (Saddle Point)**.** Let $f : U \to \mathbb{R}$ a function defined on open set $U \subseteq \mathbb{R}^n$. Suppose f is continuously differentiable over U and $\mathbf{x}^*$ a stationary point. Then $\mathbf{x}^*$ a *saddle point* if it is neither a local minimum nor a local maximum.

Theorem 2.8 (Sufficient Condition for Saddle Points)**.** *Let $f : U \to \mathbb{R}$ a function defined on open set $U \subseteq \mathbb{R}^n$. Suppose f is twice continuously differentiable over U and $\mathbf{x}^*$ a stationary point. Then:*

1. *if $\nabla f(\mathbf{x}^*) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}^*)$ is indefinite, then $\mathbf{x}^*$ a saddle point*

2.5 Attainment of Minimal/Maximal Values

Theorem 2.9 (Weierstrass Theorem)**.** *Let $f : U \to \mathbb{R}$ a continuous function defined on a non-empty compact set $U \subseteq \mathbb{R}^n$. Then f attains its maximal and minimal values over U.*

Definition 2.10 (Coerciveness)**.** A function $f : U \to \mathbb{R}$ is *coercive* if for any sequence $\{\mathbf{x}_k\} \subseteq U$ such that $\|\mathbf{x}_k\| \to \infty$, we have that $f(\mathbf{x}_k) \to \infty$.

Theorem 2.11 (Attainment of Global Optima Points for Coercive Functions)**.** *Let $f : U \to \mathbb{R}$ a coercive function defined on a non-empty closed set $U \subseteq \mathbb{R}^n$. Then f attains its global minimal value over U.*

2.6 Global Optimality Conditions

Theorem 2.12 (Global Optimality Conditions)**.** *Let f be twice continuously differentiable over $\mathbb{R}^n$. Suppose that $\nabla^2 f(x) \succeq 0$ for any $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x}^* \in \mathbb{R}^n$ be a stationary point of f. Then $\mathbf{x}^*$ is a global minimum point of f.*

2.7 Quadratic Functions

Proposition 2.13. *Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + \mathbf{c}$ be a quadratic function defined on $\mathbb{R}^n$. With $A \in \mathbb{R}^{n\times n}$ symmetric, $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}$. Then:*

1. *$\mathbf{x}$ a stationary point of f if and only if $A\mathbf{x} = -\mathbf{b}$*
2. *if $A \succeq 0$, then $\mathbf{x}$ a global minimum point of f if and only if $A\mathbf{x} = -\mathbf{b}$*
3. *if $A \succ 0$, then $\mathbf{x} = -A^{-1}\mathbf{b}$ a strict global minimum point of f*

2.8 Two Important Theorems on Quadratic Functions

Lemma 2.14 (Coerciveness of Quadratic Functions)**.** *Let* $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + \mathbf{c}$ *be a quadratic function defined on* $\mathbb{R}^n$*. With* $A \in \mathbb{R}^{n\times n}$ *symmetric,* $\mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}$*. Then* f *is coercive* $\iff A \succ 0$

Theorem 2.15 (Characterization of the Nonnegativity of Quadratic Functions)**.** *Consider the quadratic function* $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + \mathbf{c}$ *defined on* $\mathbb{R}^n$*. With* $A \in \mathbb{R}^{n\times n}$ *symmetric,* $\mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}$*. Then the following are equivalent:*

1. $f(\mathbf{x}) \equiv \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + \mathbf{c}$
2. *The augmented matrix* $\begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^T & \mathbf{c} \end{pmatrix} \succeq 0$ *is positive semidefinite*

2.9 Appendix: Classification of Matrices

Definition 2.16 (Positive Definiteness)**.** A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite* ($A \succeq 0$) if

$$\mathbf{x}^T A\mathbf{x} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$$

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive definite* ($A \succ 0$) if

$$\mathbf{x}^T A\mathbf{x} > 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$$

Proposition 2.17. *Let* A *be positive definite (semidefinite) matrix. Then the diagonal elements of* A *are positive. (non-negative)*

Definition 2.18 (Negative Definiteness)**.** A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *negative semidefinite* ($A \preceq 0$) if

$$\mathbf{x}^T A\mathbf{x} \leq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$$

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *negative definite* ($A \prec 0$) if

$$\mathbf{x}^T A\mathbf{x} < 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$$

Remark.

- A is negative (semi)definite $\iff -A$ is positive (semi)definite
- A matrix is indefinite if and only if it is neither positive nor negative semidefinite
- A symmetric matrix with positive and negative diagonal elements is indefinite
- The sum of two positive/negative (semi)definite matrices is positive/negative (semi)definite

Theorem 2.19 (Eigenvalue Characterization)**.** *Let* A *be a symmetric* $n \times n$ *matrix. Then:*

1. A *is positive definite* $\iff \lambda_i > 0, \quad \forall i = 1, 2, \ldots, n$

2. *A is positive semidefinite* $\iff \lambda_i \geq 0, \quad \forall i = 1, 2, \ldots, n$

3. *A is negative definite* $\iff \lambda_i < 0, \quad \forall i = 1, 2, \ldots, n$

4. *A is negative semidefinite* $\iff \lambda_i \leq 0, \quad \forall i = 1, 2, \ldots, n$

5. *A is indefinite* $\iff \exists i, j$ *s.t.* $\lambda_i > 0, \lambda_j < 0$

Definition 2.20 (Principal Minor)**.** Let A be a symmetric $n \times n$ matrix. The k-th *principal minor* of A is the determinant of the $k \times k$ submatrix of A obtained by deleting the last $n - k$ rows and columns of A.

Proposition 2.21. *Let A a $n \times n$ symmetric matrix. Then A is positive definite if and only if all the principal minors of A are positive.*

$$D_1(A) > 0, D_2(A) > 0, \ldots, D_n(A) > 0$$

Proposition 2.22. *Let A a $n \times n$ symmetric matrix. Then A is negative definite if and only if the principal minors of A alternate in sign, starting with a negative minor.*

$$(-1)^k D_k(A) > 0, \quad \forall k = 1, 2, \ldots, n$$

Definition 2.23 (Diagonal Dominance)**.** Let A symmetric $n \times n$ matrix

1. A is *diagonally dominant* if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n$$

2. A is *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n$$

Theorem 2.24 (Positive Definiteness of diagonally dominant matrices)**.** *1. if A symmetric, diagonally dominant, with non-negative diagonal elements, then A is positive semidefinite*

2. if A symmetric, strictly diagonally dominant, with positive diagonal elements, then A is positive definite

3 Linear and Nonlinear Least Squares Problems

3.1 Linear Least Squares

Definition 3.1 (Linear Least Squares)**.** Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. We assume A has full column rank. The *linear least squares problem* is to find $\mathbf{x}^* \in \mathbb{R}^n$ that minimizes the residual sum of squares:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|^2$$

Which is the same as:

$$\min_{x\in\mathbb{R}^n} \{f(\mathbf{x}) \equiv \mathbf{x}^T A^T A\mathbf{x} - 2\mathbf{b}^T A\mathbf{x} + \mathbf{b}^T\mathbf{b}\}$$

Note that $\nabla^2 f(x) = 2A^T A \succ 0$ since A of full column rank, and $m > n$. Therefore we get the unique optimal solution $\mathbf{x}_{LS} = \nabla f(\mathbf{x}) = 0$

$$(A^T A)\mathbf{x}_{LS} = A^T\mathbf{b}$$

$$\mathbf{x}_{LS} = (A^T A)^{-1} A^T\mathbf{b}$$

Definition 3.2 (Regularised Least Squares)**.** Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. We assume A has full column rank. The *regularised least squares problem* is to find $\mathbf{x}^* \in \mathbb{R}^n$ that minimizes the regularised residual sum of squares:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|^2 + \lambda R(x)$$

Where λ the regularisation parameter and $R(x)$ the regularisation/penalty function. A common choice is the quadratic regularisation function:

$$\min \|A\mathbf{x} - \mathbf{b}\|^2 + \lambda \|D\mathbf{x}\|^2$$

The optimal solution is then given by:

$$X_{RLS} = (A^T A + \lambda D^T D)^{-1} A^T\mathbf{b}$$

Must have that $\text{null}(S) \cap \text{null}(A) = \{0\}$ for the above inversion to be possible.

Example 3.3 (Denoising)**.** Suppose we have a noisy signal $\mathbf{y} \in \mathbb{R}^n$ that is the sum of a clean signal $\mathbf{x} \in \mathbb{R}^n$ and some noise $\mathbf{e} \in \mathbb{R}^n$. We can model this as:

$$\mathbf{y} = \mathbf{x} + \mathbf{e}$$

We can then solve the regularised least squares problem to recover the clean signal $\mathbf{x}$ from the noisy signal $\mathbf{y}$. Taking our problem as follows:

$$\min \|\mathbf{y} - \mathbf{x}\|^2 + \lambda \|L\mathbf{x}\|^2$$

For a matrix L

$$L = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

This is a common choice for denoising as it penalises the difference between adjacent elements of the signal. Direct solution is given by:

$$\mathbf{x}_{RLS} = (I + \lambda L^T L)^{-1}\mathbf{y}$$

3.2 Nonlinear Least Squares

Definition 3.4 (Nonlinear Least Squares)**.** Aim to find $\mathbf{x}^* \in \mathbb{R}^n$ that minimizes:

$$\min_x \sum_{i=1}^{m} (f_i(x) - b_i)^2$$

4 Gradient Descent Algorithm

Definition 4.1 (Descent Direction)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ a continuously differentiable function over $\mathbb{R}^n$. A vector $\mathbf{d} \in \mathbb{R}^n$ is a *descent direction* at $\mathbf{x} \in \mathbb{R}^n$ if:

$$\nabla f(\mathbf{x})^T \mathbf{d} < 0$$

Lemma 4.2. *Let f be a continuously differentiable function over $\mathbb{R}^n$, and let $\mathbf{x} \in \mathbb{R}^n$. Suppose $\mathbf{d}$ a descent direction of f at $\mathbf{x}$. Then there exists $\epsilon > 0$, such that for any $\alpha \in (0, \epsilon)$, we have:*

$$f(\mathbf{x} + \alpha\mathbf{d}) < f(\mathbf{x})$$

Algorithm 1: Schematic Descent Direction Method

Initialisation: Choose $\mathbf{x}^{(0)} \in \mathbb{R}^n$, set $k = 0$
General Step: For $k = 0, 1, 2, \ldots$ execute the following:

1. Pick a descent direction $\mathbf{d}^{(k)}$ at $\mathbf{x}^{(k)}$
2. Find a stepsize t^k satisfying $f(x^k + t^k d^k) < f(x^k)$
3. Set $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + t^k \mathbf{d}^{(k)}$
4. If a stopping criteria is satisfied, then STOP and output $\mathbf{x}^{(k+1)}$

Choosing a step-size

- **Constant stepsize:** $t^k = t$
- **Exact stepsize:** $t^k = \arg\min_t f(\mathbf{x}^{(k)} + t\mathbf{d}^{(k)})$
- **Backtracking line search:** Start with $t = 1$, and reduce t until the Armijo condition is satisfied:
$$f(\mathbf{x}^{(k)} + t\mathbf{d}^{(k)}) \leq f(\mathbf{x}^{(k)}) + \alpha t \nabla f(\mathbf{x}^{(k)})^T \mathbf{d}^{(k)}$$

Lemma 4.3. *Let f be a continuously differentiable function and let $\mathbf{x} \in \mathbb{R}^n$ be a non-stationary point. Then an optimal solution of*

$$\min_d \{f'(\mathbf{x}\delta \boldsymbol{d}) : \|\mathbf{d}\| = 1\}$$

is $\mathbf{d} = -\nabla f(\mathbf{x}) / \|\nabla f(\mathbf{x})\|$

Algorithm 2: The Gradient method

Initialisation: A tolerance parameter $\epsilon > 0$ and choose $\mathbf{x}^{(0)} \in \mathbb{R}^n$
General Step: For $k = 0, 1, 2, \ldots$ execute the following:

1. Pick a stepsize t^k by a line search method on the function
$$g(t) = f(\mathbf{x}^{(k)} - t\nabla f(\mathbf{x}^{(k)}))$$
2. Set $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - t^k \nabla f(\mathbf{x}^{(k)})$
3. If $\left\|\nabla f(\mathbf{x}^{(k+1)})\right\| < \epsilon$, then STOP and output $\mathbf{x}^{(k+1)}$

Lemma 4.4. *Let $\{x^K\}_{k>0}$ be the sequence generated by the gradient method with exact line search for solving a problem of minimizing a continuously differentiable function f. Then for any $k = 0, 1, 2, \ldots$*
$$f(\mathbf{x}^{(k+1)}) \leq f(\mathbf{x}^{(k)})$$

Definition 4.5 (Lipschitz Gradient)**.** Let f be a continuously differentiable function over $\mathbb{R}^n$. We say that f has a Lipschitz gradient if there exists $L \geq 0$ for which:

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L \|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Remark.

- If ∇f is Lipschitz with constant L, then it is also Lipschitz continuous with constant $\widetilde{L}$ for all $\widetilde{L} \geq L$
- The class of functions with Lipschitz gradient with constant L denoted by $C_L^{1,1}(\mathbb{R}^n)$ or just $C^{1,1}(\mathbb{R}^n)$

- **Linear functions:** Given $a \in \mathbb{R}^n$, the function $f(\mathbf{x}) = a^T\mathbf{x}$ has a Lipschitz gradient with constant $L = \|a\|$
- **Quadratic functions:** Given $A \in \mathbb{R}^{n\times n}$, the function $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$ a $C^{1,1}$ function with the smallest Lipschitz constant $L = 2\|A\|$

Theorem 4.6 (Equivalence to Boundedness of the Hessian)**.** *Let f be a twice continuously differentiable function over $\mathbb{R}^n$. Then the following are equivalent:*

1. *f has a Lipschitz gradient with constant L*
2. *The Hessian of f is bounded, i.e. there exists $M \geq 0$ for which:*

$$\left\|\nabla^2 f(\mathbf{x})\right\| \leq M, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

Lemma 4.7 (Sufficient decrease of the gradient method)**.** *Let $f \in C_L^{1,1}(\mathbb{R}^n)$. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the gradient method for solving*

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$$

with one of the following stepsize strategies:

- *constant stepsize $\widetilde{t} \in (0, \frac{2}{L})$*
- *exact line search*
- *backtracking procedure with parameters $s \in \mathbb{R}_{++}, \alpha \in (0,1)$ and $\beta \in (0,1)$. Then*

$$f(x^k) - f(x^{k+1}) \geq M \left\|\nabla f(x^k)\right\|^2$$

Where

$$M = \begin{cases} \widetilde{t}(1 - \widetilde{t}\frac{L}{2}) & \text{if constant stepsize} \\ \frac{1}{2L} & \text{if exact line search} \\ \alpha \min\{s, \frac{2(1-\alpha)\beta}{L}\} & \text{if backtracking line search} \end{cases}$$

Theorem 4.8 (Convergence of the Gradient Method)**.** *Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the gradient method for solving*

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$$

with one of the following stepsize strategies:

- *constant stepsize $\widetilde{t} \in (0, \frac{2}{L})$*

- *exact line search*
- *backtracking procedure with parameters* $s \in \mathbb{R}_{++}, \alpha \in (0,1)$ *and* $\beta \in (0,1)$

Assume that

- $f \in C_L^{1,1}(\mathbb{R}^n)$
- f *bounded below over* $\mathbb{R}^n$*, that is, there exists* $m \in \mathbb{R}$ *such that* $f(\mathbf{x}) > m$ *for all* $\mathbf{x} \in \mathbb{R}^n$

Then:

1. *for any* k*,* $f(x^{k+1} < f(x^k))$ *unless* $\nabla f(x^k) = 0$
2. $\nabla f(x^k) \to 0$ *as* $k \to \infty$

Definition 4.9 (Condition number)**.** Let A be an $n \times n$ positive definite matrix. The *condition number* of A is defined as:

$$\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the largest and smallest eigenvalues of A respectively.

Lemma 4.10 (Kantorovich Inequality)**.** *Let* A *be an* $n \times n$ *positive definite matrix. Then for any* $0 \neq \mathbf{x} \in \mathbb{R}^n$*, we have:*

$$\frac{(\mathbf{x}^T x)^2}{(\mathbf{x}^T A\mathbf{x})(\mathbf{x}^T A^{-1}\mathbf{x})} \leq \frac{4\lambda_{max}(A)\lambda_{min}(A)}{(\lambda_{max}(A) + \lambda_{min}(A))^2}$$

Theorem 4.11 (Gradient method for minimizing $\mathbf{x}^T A\mathbf{x}$)**.** *Let* $\{x^k\}_{k\geq 0}$ *be the sequence generated by the gradient method with exact line search for solving the problem*

$$\min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{x}^T A\mathbf{x} \quad (A \succ 0)$$

Then for any $k = 0, 1, \ldots$

$$f(x^{k+1}) \leq \left(\frac{M-m}{M+m}\right)^2 f(x^k)$$

where $M = \lambda_{max}(A)$ *and* $m = \lambda_{min}(A)$

Algorithm 3: Scaled Gradient Method

Initialisation: A tolerance parameter $\epsilon > 0$ and choose $\mathbf{x}^{(0)} \in \mathbb{R}^n$
General Step: For $k = 0, 1, 2, \dots$ execute the following:

1. Pick $\mathbf{D}^k \succ 0$, a scaling matrix
2. Pick a stepsize t^k by a line search method on the function

$$g(t) = f(\mathbf{x}^{(k)} - t\mathbf{D}^{(k)} \nabla f(x^k))$$

3. Set $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - t^k \mathbf{D}^{(k)} \nabla f(x^k)$
4. If $\left\| \nabla f(x^{k+1}) \right\| \leq \epsilon$, then STOP and output $\mathbf{x}^{(k+1)}$

Algorithm 4: The Damped Gauss-Newton Method

Initialisation: A tolerance parameter $\epsilon > 0$ and choose $\mathbf{x}^{(0)} \in \mathbb{R}^n$
General Step: For $k = 0, 1, 2, \dots$ execute the following:

1. Set $\mathbf{d}^k = \left(J(\mathbf{x}^k)^T J(\mathbf{x}^k) \right)^{-1} J(\mathbf{x}^k)^T F(\mathbf{x}^k)$
2. Set t^k by line search procedure on function

$$h(t) = g(x^k + t\mathbf{d}^k)$$

3. Set $\mathbf{x}^{k+1} = \mathbf{x}^k + t^k \mathbf{d}^k$
4. If $\left\| \nabla g(x^{k+1}) \right\| \leq \epsilon$, then STOP and output $\mathbf{x}^{(k+1)}$

5 Stochastic Gradient Descent

5.1 The Kaczmarz Algorithm

Definition 5.1 (Kaczmarz Algorithm)**.** Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The *Kaczmarz Algorithm* is an iterative method for solving the linear system $A\mathbf{x} = \mathbf{b}$. The algorithm is as follows:

1. Start with an initial guess $\mathbf{x}^{(0)} \in \mathbb{R}^n$
2. For $k = 0, 1, 2, \dots$ execute the following:
   (a) For $i = 1, 2, \dots, m$ execute the following:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{b_i - a_i^T \mathbf{x}^{(k)}}{\|a_i\|^2} a_i$$

Note that we do not need to compute A^{-1}. The i-th row that is chosen at the k-th iteration of the algorithm is cycled periodically through all m rows of A.

$$i = k \mod m$$

Provided the system is consistent, the Kaczmarz algorithm converges to the solution of the linear system.
The Kaczmarz algorithm converges exponential to the solution of the linear system, if at the k-th iteration the i-th row is chosen randomly according to either a uniform distribution or proportional to the squared row norm: $p_i = \|a_i\|^2 / \|A\|^2$,

5.2 Stochastic Gradient Descent

Definition 5.2 (Stochastic Gradient Descent (SGD))**.** An iterative optimization algorithm used in machine learning to minimize a loss function $L(\theta)$, where θ represents the parameters of the model. SGD modifies the parameters by following these steps:

1. Initialize the parameters θ to some starting values θ_0.
2. At each iteration t, randomly select a minibatch $\mathcal{B}_t$ from the dataset.
3. Compute the gradient of the loss function approximated over the minibatch:

$$g_t = \frac{1}{|\mathcal{B}_t|} \sum_{i \in \mathcal{B}_t} \nabla L_i(\theta_t)$$

   where $\nabla L_i(\theta_t)$ is the gradient of the loss function with respect to θ computed at the i-th data point.

4. Update the parameters using a learning rate η:

$$\theta_{t+1} = \theta_t - \eta g_t$$

5. Repeat steps 2-4 until the parameters converge or a predefined number of iterations is reached.

SGD is particularly effective for large datasets as it requires less computational resources per iteration. The random selection of data helps in avoiding local minima, but it also introduces variability in the gradient estimation, potentially causing fluctuation in the convergence path.

Theorem 5.3 (Convergence of SGD)**.** *Assume:*

- *The cost $g(\mathbf{x})$ is such that*

$$\|\nabla g(\mathbf{x}) - \nabla g(\mathbf{y})\| \leq L \|\mathbf{x} - \mathbf{y}\|, \quad \textit{and } \nabla^2 g(\mathbf{x}) \succeq \mu I$$

- *The sample gradient* $\nabla Q_i(\mathbf{x}^k)$ *is an unbiased estimator of* $\nabla g(\mathbf{x}^k)$
- *For all* $\mathbf{x}$
$$\mathbb{E}\left[\|Q_i(\mathbf{x})\|^2\right] \leq \sigma^2 + c\,\|\nabla g(x)\|^2$$

  Then if $t^k \equiv t \leq \frac{1}{Lc}$ *Then SGD achieves*
$$\mathbb{E}\left[g(\mathbf{x}^k) - g(\mathbf{x}^*)\right] \leq \frac{tL\sigma^2}{2\mu} + (1-\mu)^k\left[g(\mathbf{x}^0) - g(\mathbf{x}^*)\right]$$

The above implies

- *Fast (linear) convergence during first iterations*
- *Convergence to a neighbourhood of the optimal solution, without further improvement*
- *If gradient computation noiseless, that is* $\sigma = 0$*, then linear convergence to the optimal solution*
- *A smaller stepsize* t *yield better converging points*

6 Convex Sets and Functions

Definition 6.1 (Convex Set)**.** A set $C \subseteq \mathbb{R}^n$ is *convex* if for any $\mathbf{x}, \mathbf{y} \in C$ and any $\lambda \in [0, 1]$, we have:
$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$$

Example 6.2.

- Any line segment in $\mathbb{R}^n$ is a convex set
$$L = \{\mathbf{z} + t\mathbf{d} : t \in \mathbb{R}\}$$

  where $\mathbf{z}, 0 \neq \mathbf{d} \in \mathbb{R}^n$
- $[x, y], (x, y)$ for $x, y \in \mathbb{R}^n, x \neq y, \varnothing, \mathbb{R}^n$
- A hyperplane is a convex set
$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$$
- A half-space is a convex set
$$H^- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leq b\}$$
- The open and closed balls are convex sets

- An ellipsoid is a convex set

$$E = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T Q \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + c \le 0\}, \quad Q \succ 0$$

Lemma 6.3 (Intersection of Convex Sets)**.** *The intersection of any collection of convex sets is a convex set*

Theorem 6.4 (Properties of Convex Sets)**.** *1. Let $C_1, C_2, \dots, C_k \subseteq \mathbb{R}^n$ be convex sets and let $\mu_1, \mu_2, \dots, \mu_k \in \mathbb{R}$. Then the set*

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in \mu_1 C_1 + \mu_2 C_2 + \ldots + \mu_k C_k\}, \quad \textit{is convex}$$

2. Let $C_i \subset \mathbb{R}^{k_i}, i = 1, \dots, m$ be convex sets. Then

$$C = C_1 \times C_2 \times \ldots \times C_m = \{(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) : \mathbf{x}_i \in C_i, i = 1, \dots, m\}, \quad \textit{is convex}$$

3. Let $M \subseteq \mathbb{R}^n$ be a convex set and let $A \in \mathbb{R}^{m \times n}$. Then the set

$$A(M) = \{A\mathbf{x} : \mathbf{x} \in M\}, \quad \textit{is convex}$$

4. Let $D \subseteq \mathbb{R}^m$ be convex and let $A \in R^{m \times n}$. Then the set

$$A^{-1}(D) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \in D\}, \quad \textit{is convex}$$

Definition 6.5 (Convex Combinations)**.** Given m points $x_1, x_2, \dots, x_m \in \mathbb{R}^n$, a convex combination of these points is a point of the form:

$$\sum_{i=1}^{m} \lambda_i x_i, \quad \text{where } \lambda_i \ge 0, \sum_{i=1}^{m} \lambda_i = 1$$

Theorem 6.6. *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $x_1, x_2, \dots, x_m \in C$. Then for any $\lambda \in \Delta_m$, the relation $\sum_{i=1}^m \lambda_i \mathbf{x}_i \in C$ holds*

Definition 6.7 (The Convex Hull)**.** Let $S \subseteq \mathbb{R}^n$. The convex hull of S, denoted by $\text{conv}(S)$, is the set of all convex combinations of points in S:

$$\text{conv}(S) = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in S, \lambda_i \ge 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

Theorem 6.8 (Caratheodory)**.** *Let $S \subseteq \mathbb{R}^n$ and let $\mathbf{x} \in conv(S)$. Then there exists a subset $S' \subseteq S$ with $|S'| \le n+1$ such that $\mathbf{x} \in conv(S')$. That is there exists $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1} \in S$ such that $\mathbf{x} \in conv(\{x_1, x_2, \dots, x_{n+1}\})$, that is there exists $\lambda \in \Delta_{n+1}$ such that*

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i$$

Definition 6.9 (Extreme Point)**.** A point $\mathbf{x} \in C$ is an *extreme point* of a convex set C if it cannot be expressed as a convex combination of two distinct points in C. That is, $\mathbf{x}$ is an extreme point of C if for any $\mathbf{y}, \mathbf{z} \in C$ and any $\lambda \in (0, 1)$, we have:

$$\mathbf{x} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{z} \implies \mathbf{x} = \mathbf{y} = \mathbf{z}$$

The set of all extreme points of a convex set C is denoted by $\text{ext}(C)$.

Theorem 6.10 (The Krein-Milman Theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be a compact convex set. Then*

$$S = conv(ext(S))$$

6.1 Convex Functions

Definition 6.11 (Convex Function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, we have:

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$$

Definition 6.12 (Strict Convexity)**.** A function $f : C \to \mathbb{R}$ defined on convex set $C \subseteq \mathbb{R}^n$ is called strictly convex if for any $\mathbf{x}, \mathbf{y} \in C$ and any $\lambda \in (0, 1)$, we have:

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$$

Definition 6.13 (Concavity)**.** A function f is concave if $-f$ is convex. Equivalently; a function $f : \mathbb{R}^n \to \mathbb{R}$ is *concave* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, we have:

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$$

Theorem 6.14 (Jensen's Inequality)**.** *Let $f : C \to \mathbb{R}$ be a convex function where $C \subseteq \mathbb{R}^n$ is a convex set. Then for any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in C$ and any $\lambda \in \Delta_m$, we have:*

$$f\left(\sum_{i=1}^{m} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{m} \lambda_i f(\mathbf{x}_i)$$

Definition 6.15 (Epigraph)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The *epigraph* of f is the set:

$$\text{epi}(f) = \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : f(\mathbf{x}) \leq t\}$$

Theorem 6.16. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then f is convex if and only if $epi(f)$ is a convex set*

6.2 First-order Characterizations of Convexity

Theorem 6.17 (Gradient Inequality)**.** *Let $f : C \to \mathbb{R}$ be a continuously differentiable function defined on a convex set $C \subseteq \mathbb{R}^n$. Then f is convex if and only if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}), \quad \forall \mathbf{x}, \mathbf{y} \in C$$

Theorem 6.18 (Stationarity Implies Global Optimality)**.** *Let f be a continuously differentiable function convex over a convex set $C \subseteq \mathbb{R}^n$. Suppose $\nabla f(\mathbf{x}^*) = 0$ for some $\mathbf{x}^* \in C$. Then $\mathbf{x}^* \in C$. Then $\mathbf{x}^*$ is a global minimizer of f over C*

Theorem 6.19 (Convexity of Quadratic Functions with Positive semidefinite matrices)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function; $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, $_bf \in \mathbb{R}^n, c \in \mathbb{R}$. Then f is (strictly) convex if and only if $A \succeq 0$ ($A \succ 0$)*

Theorem 6.20 (Monotonicity of the Gradient)**.** *Suppose that f is a continuously differentiable function over a convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if*

$$\nabla f(\mathbf{y}) - \nabla f(\mathbf{x}) \succeq 0, \quad \forall \mathbf{x}, \mathbf{y} \in C$$

6.3 Second-order Characterizations of Convexity

Theorem 6.21 (Second-order Characterization of Convexity)**.** *Let f be a twice continuously differentiable function over an open convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if $\nabla^2 \succeq 0$ for any $\mathbf{x} \in C$*

6.4 Further Results for Convex Functions

Lemma 6.22 (Operations Preserving Convexity)**.** *The following preserve convexity:*

- *Let f be a convex function over a convex set $C \subseteq \mathbb{R}^n$. Then the αf is convex over C for any $\alpha \geq 0$*
- *Let f and g be convex functions over a convex set $C \subseteq \mathbb{R}^n$. Then the function $f + g$ is convex over C*
- *Let f be a convex function over convex set $C \subseteq \mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the function g defined by*

$$g(\mathbf{y}) = f(A\mathbf{y} + \mathbf{b}) \text{ is convex over } D = \{\mathbf{y} \in \mathbb{R}^m : A\mathbf{y} + \mathbf{b} \in C\}$$

Theorem 6.23 (Preservation of Convexity Under Partial Minimization)**.** *Let $f : C \times D \to \mathbb{R}$ be a convex function over the set $C \times D$ where $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ are convex sets. Let*

$$g(\mathbf{x}) = \min_{\mathbf{y} \in D} f(\mathbf{x}, \mathbf{y}), \quad x \in C$$

where we assume that the minimum is finite. Then g is convex over C

Definition 6.24 (Level sets)**.** Let $f : S \to \mathbb{R}$ be a function defined over a set $S \subseteq \mathbb{R}^n$. The level set of f at level t is the set:

$$\text{Lev}(f, \alpha) = \{\mathbf{x} \in S : f(\mathbf{x}) \leq \alpha\}$$

Theorem 6.25 (Convexity of Level Sets)**.** *Let $f : S \to \mathbb{R}$ be a convex function defined over a convex set $S \subseteq \mathbb{R}^n$. Then the level set of f at level t is a convex set for any $t \in \mathbb{R}$*

6.4.1 Four Important Theorems for Convex Functions

Theorem 6.26 (Continuity of Convex Functions)**.** *Let $f : S \to \mathbb{R}$ be a convex function defined over a convex set $S \subseteq \mathbb{R}^n$. Then f is continuous over S. In particular let $\mathbf{x}_0 \in$ int(S). Then there exists $\epsilon > 0$ and $L > 0$ such that $B(\mathbf{x}_0, \epsilon) \subseteq C$ and*

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq L \left\|\mathbf{x} - \mathbf{x}_0\right\|, \quad \forall \mathbf{x} \in B(\mathbf{x}_0, \epsilon)$$

Theorem 6.27 (Existence of directional derivatives of Convex Functions)**.** *Let $f : S \to \mathbb{R}$ be a convex function defined over a convex set $S \subseteq \mathbb{R}^n$. Then for any $\mathbf{x} \in$ int(S) and any $0 \neq \mathbf{d} \in \mathbb{R}^n$, the directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{d}$ exists and is given by:*

$$f'(\mathbf{x}; \mathbf{d}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x})}{t}$$

Theorem 6.28 (No Maximum inside the Convex Set)**.** *Let $f : C \to \mathbb{R}$ be a convex function defined over a non-empty convex set $C \subseteq \mathbb{R}^n$. Then f does not attain a maximum at a point in int(C)*

Theorem 6.29 (Maximimum of a Convex Function over a compact convex set)**.** *Let $f : C \to \mathbb{R}$ be convex over the nonempty convex and compact set $C \subseteq \mathbb{R}^n$. Then there exists at least one maximiser of f over C that is an extreme point of C*

7 Convex Optimisation

Theorem 7.1 (Local minima are global in CVX)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function defined over a convex set $C \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}^* \in C$ is a local minimizer of f over C. Then $\mathbf{x}^*$ is a global minimum of f over C*

Theorem 7.2. *Let $f : C \to \mathbb{R}$ be a strictly convex function defined on the convex set C. Let $\mathbf{x}^* \in C$ be a local minimum of f over C. Then $\mathbf{x}^*$ is a strict global minimum of f over C*

Theorem 7.3. *Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Then the set of optimal solutions of the problem*

$$\min\{f(\mathbf{x}) : \mathbf{x} \in C\}$$

is a convex set. If also f is strictly convex, then the set of optimal solutions is a singleton

Definition 7.4 (Stationarity)**.** Let f be a continuously differentiable function over a closed and convex set C. Then $\mathbf{x}^*$ is called a stationary point of $\min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{x} \in C\}$ if

$$\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in C$$

Theorem 7.5 (Stationarity as a Necessary Optimality Condition)**.** *Let f be a continuously differentiable function over a nonempty closed convex set C, and let $\mathbf{x}^*$ be a local minimum of $\min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{x} \in C\}$. Then $\mathbf{x}^*$ is a stationary point of the problem.*

7.1 The Orthogonal Projection Operator

Definition 7.6 (Orthogonal Projection)**.** Given a nonempty closed convex set C, the orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by:

$$P_C(\mathbf{x}) = \arg\min_{\mathbf{y} \in C}\{\|\mathbf{x} - \mathbf{y}\|^2 : \mathbf{y} \in C\}$$

Theorem 7.7 (The First Projection Theorem)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed and convex set. Then for any $\mathbf{x} \in \mathbb{R}^n$, the orthogonal projection $P_C(\mathbf{x})$ exists and is unique*

Theorem 7.8 (The Second Projection Theorem)**.** *Let C be a nonempty closed convex set and let $\mathbf{x} \in \mathbb{R}^n$. Then $z = P_C(\mathbf{x})$ if and only if*

$$(\mathbf{x} - \mathbf{z})^T(\mathbf{y} - \mathbf{z}) \leq 0, \quad \forall \mathbf{y} \in C$$

Theorem 7.9 (Reperesentation of Stationarity via the Orthogonal Projection Operator)**.** *Let f be a continuously differentiable function over the nonempty closed convex set C, and let $s > 0$. Then $\mathbf{x}^*$ is a stationarity point of the problem $\min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{x} \in C\}$ if and only if*

$$\mathbf{x}^* = P_C(\mathbf{x}^* - s\nabla f(\mathbf{x}^*))$$

7.2 The Gradient Projection Method

Algorithm 7: The Gradient Projection Method

Initialisation: A tolerance parameter $\epsilon > 0$ and choose $\mathbf{x}^{(0)} \in \mathbb{R}^n$
General Step: For $k = 0, 1, 2, \ldots$ execute the following:

1. Pick stepsize t^k by a line search procedure.
2. Set $\mathbf{x}^{(k+1)} = P_C(\mathbf{x}^{(k)} - t^k\nabla f(\mathbf{x}^{(k)}))$
3. If $\left\|\mathbf{x}^k - \mathbf{x}^{k+1}\right\| \leq \epsilon$, then STOP and output $\mathbf{x}^{(k+1)}$

Algorithm 8: The Gradient Projection Method with Backtracking

Initialisation: A tolerance parameter $\epsilon > 0$ and choose $\mathbf{x}^{(0)} \in \mathbb{R}^n$, Parameters: $s > 0, \alpha \in (0,1)$ and $\beta \in (0,1)$ **General Step:** For $k = 0, 1, 2, \ldots$ execute the following:

1. Set $t^k = s$
2. While $f(x^k) - f(P_C(\mathbf{x}f - t^k \nabla f(\mathbf{x}^k))) < \alpha t^k \left\| G_{\frac{1}{t^k}} \right\|^2$, set $t^k = \beta t^k$
3. Set $\mathbf{x}^{(k+1)} = P_C(\mathbf{x}^{(k)} - t^k \nabla f(\mathbf{x}^{(k)}))$
4. If $\left\| \mathbf{x}^k - \mathbf{x}^{k+1} \right\| \leq \epsilon$, then STOP and output $\mathbf{x}^{(k+1)}$

Theorem 7.10 (Convergence of the Gradient Projection Method)**.** *Let $\{\mathbf{x}^k\}$ be the sequence generated by the gradient projection method for solving problem $\min_{\mathbf{x} \in C}\{f(\mathbf{x})\}$ with either a constant stepsize $\bar{t} \in (0, \frac{2}{L})$, where L a Lipschitz constant of ∇f or a backtracking stepsize strategy. Assume f bounded below. Then:*

1. *The sequence $\{f(\mathbf{x}^k)\}$ is nonincreasing*
2. *$G_d(\mathbf{x}^k) \to 0$ as $k \to \infty$, where*

$$d = \begin{cases} \frac{1}{\bar{t}} & \text{if constant stepsize} \\ \frac{1}{s} & \text{if backtracking} \end{cases}$$

8 Optimality Conditions

8.1 Separation Theorem

Definition 8.1. A hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T \mathbf{x} = b\} \quad \text{where } \mathbf{a} \in \mathbb{R}^n \setminus \{0\}, b \in \mathbb{R}$$

is said to strictly separate a point $\mathbf{y} \notin S$ from S if

$$\mathbf{a}^T \mathbf{y} > b$$

and

$$\mathbf{a}^T \mathbf{x} \leq b \quad \forall \mathbf{y} \in S$$

Theorem 8.2 (Separation of a Point from a Closed and Convex Set)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed and convex set, and let $\mathbf{y} \notin C$. Then there exists $\mathbf{p} \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\mathbf{p}^T \mathbf{y} > \alpha$ and $\mathbf{p}^T \mathbf{x} \leq \alpha$ for all $\mathbf{x} \in C$*

Lemma 8.3 (Farkas' Lemma - an Alternative Theorem)**.** *Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then exactly one of the following systems has a solution:*

1. $\mathbf{Ax} \leq 0, \mathbf{c}^T\mathbf{x} > 0$
2. $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq 0$

Lemma 8.4 (Farkas' Lemma - Second Formulation)**.** *Let $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then teh following two claims are equivalent:*

1. *The implication $\mathbf{Ax} \leq 0 \implies \mathbf{c}^T\mathbf{x} \leq 0$ holds true*
2. *There exists $\mathbf{y} \in \mathbb{R}^m_+$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{c}$*

Theorem 8.5 (Gordan's Alternative Theorem)**.** *Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then exactly one of the follwing systems has a solution:*

1. $\mathbf{Ax} < 0$
2. $\mathbf{p} \neq \mathbf{0}, \mathbf{A}^T\mathbf{p} = 0, \mathbf{p} \geq 0$

Theorem 8.6 (KKT conditions for Linearly Constrained Problems - Necessary Optimality Conditions)**.** *Consider minimization problem*

$$\begin{cases} \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \\ \textit{subject to } \mathbf{Ax} = \mathbf{b} \end{cases}$$

where f is continuously differentiable over $\mathbf{a}_1, \ldots, \mathbf{a}_n,\ b_1, \ldots, b_m \in \mathbb{R}$ and let $\mathbf{x}^*$ be a local minimum point of the problem. Then there exists $\lambda_1, \ldots \lambda_m \geq 0$ such that:

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \mathbf{a}_i = 0$$

and

$$\lambda_i(\mathbf{a}_i^T \mathbf{x}^* - b_i) = 0, \quad i = 1, \ldots, m$$

Theorem 8.7 (KKT Conditions for Convex Linearly Constrained Problems - Necessary and Sufficient Optimality Conditions)**.** *Consider the minimization problem*

$$\begin{cases} \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \\ \textit{subject to } \mathbf{Ax} = \mathbf{b} \end{cases}$$

where in addition f is a convex continuously differentiable function over $\mathbb{R}^n$, and let $\mathbf{x}^*$ be a feasible solution. Then $\mathbf{x}^*$ is an optimal solution if and only if there exists $\lambda_1, \ldots, \lambda_m \geq 0$ such that:

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \mathbf{a}_i = 0$$

and

$$\lambda_i(\mathbf{a}_i^T \mathbf{x}^* - b_i) = 0, \quad i = 1, \ldots, m$$

Theorem 8.8 (KKT Conditions for Linearly Constrained Problems)**.** *Consider minimization problem*

$$\begin{cases} & \min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \\ \textit{subject to} & \mathbf{a}_i^T\mathbf{x} \le b_i, \quad i = 1,\dots,m \\ & \mathbf{c}_j^T\mathbf{x} = d_j, \quad j = 1,\dots,p \end{cases}$$

where f is continuously differentiable $a_i, c_j \in \mathbb{R}^n, b_i, d_j \in \mathbb{R}$

1. *(Necessity of the KKT) If $\mathbf{x}^*$ is a local minimum of the problem, then there exist $\lambda_1,\dots,\lambda_m \ge 0$ and $\mu_1,\dots,\mu_p \in \mathbb{R}$ such that:*

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \mathbf{a}_i + \sum_{j=1}^{p} \mu_j \mathbf{c}_j = 0$$

$$\lambda_i(\mathbf{a}_i^T\mathbf{x}^* - b_i) = 0, \quad i = 1,\dots,m$$

2. *(Sufficiency of the KKT) If f is convex over $\mathbb{R}^n$ and $\mathbf{x}^*$ is a feasible solution of the problem, for which there exist $\lambda_1,\dots,\lambda_m \ge 0$ and $\mu_1,\dots,\mu_p \in \mathbb{R}$ such that the KKT conditions are satisfied, then $\mathbf{x}^*$ is an optimal solution of the problem*

8.2 Orthogonal Projections

Definition 8.9 (Orthogonal Projection onto Affine Spaces)**.** Let C be the affine space $C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by:

$$P_C(\mathbf{y}) = \mathbf{y} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{y} - \mathbf{b})$$

Definition 8.10 (Orthogonal Projection onto Hyperplanes)**.** Consider the hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\} \quad \text{where } \mathbf{a} \in \mathbb{R}^n \setminus \{0\}, b \in \mathbb{R}$$

The orthogonal projection operator $P_H : \mathbb{R}^n \to H$ is defined by:

$$P_H(\mathbf{y}) = \mathbf{y} - \frac{\mathbf{a}^T\mathbf{y} - b}{\|\mathbf{a}\|^2}\mathbf{a}$$

Lemma 8.11 (Distance of a point from a hyperplane)**.** *Let $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$ be a hyperplane where $\mathbf{a} \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. Then for any $\mathbf{y} \in \mathbb{R}^n$, the distance of $\mathbf{y}$ from H is given by:*

$$dist(\mathbf{y}, H) = \frac{\left|\mathbf{a}^T\mathbf{y} - b\right|}{\|\mathbf{a}\|}$$

Lemma 8.12. *Let $H^- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \le b\}$ be a half-space where $\mathbf{a} \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. Then*

$$P_{H^-}(\mathbf{x}) = \mathbf{x} - \frac{[\mathbf{a}^T\mathbf{x} - b]_+}{\|\mathbf{a}\|^2}\mathbf{a}$$

8.3 KKT Conditions for nonlinear problems

Lemma 8.13. *If* $\mathbf{x}^*$ *a local optimal solution of the following, then there are no feasible descent directions.*

$$\min f(\mathbf{x}) \quad s.t\ \mathbf{x} \in C$$

for some convex set $C \subseteq \mathbb{R}^n$*, and* f *continuously differentiable. We say* $\mathbf{d} \neq 0$ *a feasible descent direction at* $\mathbf{x} \in C$ *if*

1. $\nabla f(\mathbf{x})^T \mathbf{d} < 0$*, and*
2. *There exists* $\epsilon > 0$ *such that* $\mathbf{x} + t\mathbf{d} \in C$ *for all* $t \in (0, \epsilon)$

Extending this to the problem:

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t & g_i(\mathbf{x}) \leq 0 \quad i = 1, \ldots, m \end{cases}$$

We introduce the notion of ***active constraints****. We say the* i*-the constraint is active at* $\widetilde{\mathbf{x}}$ *if* $g_i(\widetilde{\mathbf{x}}) = 0$ *i.e. when the constraints are binding.*

$$I(\widetilde{\mathbf{x}}) = \{i = \{1, \ldots, m\} : g_i(\widetilde{\mathbf{x}})\} \quad \textit{The set of active constraints at } \widetilde{\mathbf{x}}$$

Lemma 8.14. *Let* $\mathbf{x}^*$ *be a local minimum of the problem*

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t & g_i(\mathbf{x}) \leq 0 \quad i = 1, \ldots, m \end{cases}$$

where f *and* $g_1, \ldots g_m$ *are continuously differentiable functions over* $\mathbb{R}^n$*. Let* $I(\mathbf{x}^*)$ *be the set of active constraints at* $\mathbf{x}^*$*. Then, there does not exist a vector* $d \in \mathbb{R}^n$ *such that*

1. $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$*, and*
2. $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0, \quad \forall i \in I(\mathbf{x}^*)$

8.4 KKT Conditions for nonlinear convex problems

Theorem 8.15 (Sufficiency of the KKT conditions for convex optimization problems)**.** *Let* $\mathbf{x}^*$ *be a feasible solution of*

$$\begin{cases} \min & f(\mathbf{x}) \\ \textit{subject to} & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \ldots, p \end{cases}$$

where $f, g_1, \ldots, g_n$ are continuously differentiable convex functions over $\mathbb{R}^n$ and $h_1, \ldots, h_p$ are affine functions. Suppose that there exist multipliers $\lambda_1, \ldots, \lambda_m \geq 0$ and $\mu_1, \ldots, \mu_p \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}^*) = 0$$

$$\lambda_i g_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, m$$

Then $\mathbf{x}^*$ an optimal solution of the problem

Theorem 8.16 (Necessity of the KKT conditions under the generalized Slater's Condition)**.** *Let $\mathbf{x}^*$ be an optimal solution of the problem*

$$\begin{cases} & \min f(\mathbf{x}) \\ \textit{subject to} & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_j(\mathbf{x}) \leq 0, \quad j = 1, \ldots, p \\ & s_k(\mathbf{x}) = 0, \quad k = 1, \ldots, q \end{cases}$$

where $f, g_1, \ldots, g_m$ are continuously differentiable convex functions over $\mathbb{R}^n$, and $h_1, \ldots, h_p, s_1, \ldots, s_k$ are affine functions. Suppose that there exists a point $\hat{\mathbf{x}}$ satisfying the generalized Slater's condition:

$$g_i(\hat{\mathbf{x}}) < 0, \quad i = 1, \ldots, m$$
$$h_j(\hat{\mathbf{x}}) \leq 0, \quad j = 1, \ldots, p$$
$$s_k(\hat{\mathbf{x}}) = 0, \quad k = 1, \ldots, q$$

Then there exist multipliers $\lambda_1, \ldots, \lambda_m \geq 0$ and $\mu_1, \ldots, \mu_p \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \eta_j \nabla h_j(\mathbf{x}^*) + \sum_{k=1}^{q} \mu_k \nabla s_k(\mathbf{x}^*) = 0$$

$$\lambda_i g_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, m$$
$$\eta_j h_j(\mathbf{x}^*) = 0, \quad j = 1, \ldots, p$$

9 Duality

Definition 9.1 (Primal Problem)**.** The primal problem is the problem of the form:

$$\begin{cases} & \min f(\mathbf{x}) \\ \text{subject to} & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \ldots, p \\ & x \in X \end{cases}$$

where $f, g_1, \ldots, g_m$ are functions defined on the set $X \subseteq \mathbb{R}^n$. This is the "usual" optimization problem. The Lagrangian associated to this problem is

$$L(\mathbf{x}, \lambda, \eta) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}) \quad (\mathbf{x} \in X, \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p)$$

The domain of the dual objective function is

$$\text{dom}(q) = \{(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^p : q(\lambda, \mu) > -\infty\}$$

The **dual problem** is given by

$$\begin{aligned} q^* &= \max q(\lambda, \mu) \\ \text{subject to } & (\lambda, \mu) \in \text{dom}(q) \end{aligned}$$

Theorem 9.2. *Consider the primal problem with $f, g_i, h_j, i = 1, \ldots, m, j = 1, \ldots, p$, functions defined on the set $X \subseteq \mathbb{R}^n$, and let q be the dual function defined in the Dual problem. Then:*

1. *dom(q) is a convex set*
2. *q is a concave function over dom(q)*

9.1 Weak and Strong Duality

Theorem 9.3 (Weak Duality Theorem)**.** *Consider the primal problem and its dual problem. Then*

$$q^* \leq f^*$$

where f^*, q^* are the primal and dual optimal values respectively

Theorem 9.4 (Supporting Hyperplane Theorem)**.** *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $\mathbf{y} \notin C$. Then there exists $\mathbf{0} \neq \mathbf{p} \in \mathbb{R}^n$ such that*

$$\mathbf{p}^T\mathbf{x} \leq \mathbf{p}^T\mathbf{y} \quad \forall \mathbf{x} \in C$$

Theorem 9.5 (Separation of Two Convex Sets)**.** *Let $C_1, C_2, \subseteq \mathbb{R}^n$ be two nonempty convex sets such that $C_1 \cap C_2 = \emptyset$. Then there exists $\mathbf{p} \neq \mathbf{0}$ for which*

$$\mathbf{p}^T\mathbf{x} \leq \mathbf{p}^T\mathbf{y} \quad \forall \mathbf{x} \in C_1, \forall \mathbf{y} \in C_2$$

Theorem 9.6 (Nonlinear Farkas Lemma)**.** *Let $X \subseteq \mathbb{R}^n$ be a convex set and let $f, g_1, \ldots, g_m$ be convex functions defined over X. Assume that there exists $\hat{x} \in X$ such that*

$$g_1(\hat{\mathbf{x}}) < 0, g_2(\hat{\mathbf{x}}) < 0, \ldots, g_m(\hat{\mathbf{x}}) < 0$$

Let $c \in \mathbb{R}$. Then the following two claims are equivalent:

1. *The following implication holds true:*

$$\mathbf{x} \in X, g_i(\mathbf{x}) \leq 0, i = 1, \ldots, m \implies f(\mathbf{x}) \geq c$$

2. *There exists $\lambda_1, \ldots, \lambda_m \geq 0$ such that*

$$\min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} \geq c$$

Theorem 9.7 (Strong Duality of Convex Problems with Inequality Constraints)**.** *Consider the optimization problem*

$$\begin{aligned} f^* &= \min f(\mathbf{x}) \\ \textit{subject to } g_i(\mathbf{x}) &\leq 0, \quad i = 1, \ldots, m \\ \mathbf{x} &\in X \end{aligned}$$

where X is a convex set and $f, g_1, \ldots, g_m$ are convex functions defined over X. Suppose that there exists $\hat{\mathbf{x}} \in X$ for which $g_i(\hat{\mathbf{x}}) < 0, i = 1, 2, \ldots, m$. If this problem has a finite optimal value, then

1. *the optimal value of the dual problem is attained*

2. *the primal and dual problems have the same optimal value $f^* = q^*$*

Theorem 9.8 (Complementary Slackness Conditions)**.** *Consider the optimization problem,*

$$f^* := \min\{\min f(\mathbf{x}) : g_i(\mathbf{x}) \leq 0, i = 1, \ldots, m, \mathbf{x} \in X\}$$

and assume that $f^* = q^*$ where q^* is the optimal value of the dual problem. Let $\mathbf{x}^*, \lambda^*$ be feasible solutions of the primal and dual problems. Then $\mathbf{x}^*, \lambda^*$ are optimal solutions of the primal and dual problems iff

$$\begin{aligned} \mathbf{x}^* &\in \arg\min_{\mathbf{x} \in X} L(\mathbf{x}, \lambda^*) \\ \lambda_i^* g_i(\mathbf{x}^*) &= 0, \quad i = 1, \ldots, m \end{aligned}$$

Theorem 9.9 (General Strong Duality Theorem)**.** *Consider the optimization problem*

$$\begin{aligned} f^* = \min f(\mathbf{x}) \\ \textit{subject to } g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ h_j(\mathbf{x}) \leq 0, \quad j = 1, \ldots, p \\ s_k(\boldsymbol{x}) = 0, \quad k = 1, \ldots, q \\ \mathbf{x} \in X \end{aligned}$$

where X is a convex set and $f, g_i, i = 1, \ldots, m$ are convex functions over X The functions h_j, s_k are affine functions. Suppose that there exists $\hat{\mathbf{x}} \in int(X)$ for which $g_i(\hat{\mathbf{x}}) < 0, h_j(\hat{\mathbf{x}}) \leq 0$ and $s_k(\hat{\mathbf{x}}) = 0$. Then if the problem has a finite optimal value, then the optimal value of the dual problem

$$q^* = \max\{q(\lambda, \eta, \mu) : (\lambda, \eta, \mu) \in dom(q)\}$$

where

$$q(\lambda, \eta, \mu) = \min_{\mathbf{x} \in X} \left[f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \eta_j h_j(\mathbf{x}) + \sum_{k=1}^{q} \mu_k s_k(\mathbf{x}) \right]$$

is attained, and $f^* = q^*$

9.2 Three Important Examples of Duality Use

9.2.1 Linear Programming

Consider the linear programming problem

$$\begin{aligned} f^* = \min\{c^T \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\} \\ \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m \times n} \end{aligned}$$

Assume that the problem is feasible, implying strong duality holds. THe Lagrangian is given by

$$L(\mathbf{x}, \lambda) = \mathbf{c}^T \mathbf{x} + \lambda^T (\mathbf{A}\mathbf{x} - \mathbf{b}) = (\mathbf{c} + \mathbf{A}^T \lambda)^T \mathbf{x} - \mathbf{b}^T \lambda$$

and the dual objective function is

$$q(\lambda) = \min_{\mathbf{x} \in \mathbb{R}^n} L(\mathbf{x}, \lambda) = \min_{x \in \mathbb{R}^n} (\mathbf{c} + \mathbf{A}^T \lambda)^T \mathbf{x} - \mathbf{b}^{\lambda} = \begin{cases} -\mathbf{b}^T \lambda & \text{if } \mathbf{c} + \mathbf{A}^T \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, the dual problem is

$$\begin{aligned} \max -\mathbf{b}^T \lambda \\ \text{subject to } \mathbf{c} + \mathbf{A}^T \lambda = 0 \\ \lambda \geq 0 \end{aligned}$$

9.2.2 Strictly Convex Quadratic Programming

Consider the strictly convex quadratic programming problem

$$\begin{aligned} \min\ & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{f}^T\mathbf{x} \\ \text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ positive definite, $\mathbf{f} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^n$. The Lagrangian is given by

$$L(\mathbf{x}, \lambda) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{f}^T\mathbf{x} + 2\lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2(\mathbf{A}^T\lambda + \mathbf{f})^T\mathbf{x} - 2\mathbf{b}^T\lambda$$

the minimizer of the Lagrangian is attained at $\mathbf{x}^* = -\mathbf{Q}^{-1}(\mathbf{f} + \mathbf{A}^T\lambda)$. With this, we work over the dual objective,

$$\begin{aligned} q(\lambda) &= L(\mathbf{x}^*, \lambda) \\ &= -\lambda^T\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^T\lambda - 2(\mathbf{A}\mathbf{Q}^{-1}\mathbf{f} + \mathbf{b})^T\lambda - \mathbf{f}^T\mathbf{Q}^{-1}\mathbf{f} \end{aligned}$$

9.2.3 Computing the Orthogonal onto the Unit Simplex

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10 Optimal Control

10.1 What is Optimal Control?

Definition 10.1. Optimal control is the problem of finding a control function that minimizes a given cost functional, subject to a set of differential equations that describe the dynamics of the system.

We do this by means of an objective function to be optimized, i.e. as a **dynamic optimization** problem such as

$$\begin{aligned} \min_{\mathbf{u}(\cdot)\in\mathcal{U}} & \int_0^T L(\mathbf{x}(s), \mathbf{u}(s))\ ds + \Phi(\mathbf{x}(T)) \\ \text{subject to } & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ & \mathbf{x}(0) = \mathbf{x}_0 \end{aligned}$$

We have the control signal in a space of **admissible controls** $\mathcal{U}$ representing the control constraints of our problem. The running cost $L(\mathbf{x}, \mathbf{u})$ is a **running cost**, expressing our wish to achieve an objective with a certain control budget.
We have $\Phi(\mathbf{x}(T))$ a **final time penalty**, encoding the fact that when our optimization finishes at time $t = T$, we expect to find the system in the reference position. Here the

time horizon can be fixed (T or $+\infty$) or variable, i.e treated as an additional optimization variable.

Definition 10.2 (Closed-loop Optimal Control)**.** In the optimal control problem, the control signal $\mathbf{u}(t)$ is a function of the state $\mathbf{x}(t)$, i.e. $\mathbf{u} = \mathbf{u}(\mathbf{x})$. This is known as the **closed-loop** optimal control problem. The optimal control is a function of the state of the system.

Definition 10.3 (Open-loop Optimal Control)**.** In the optimal control problem, the control signal $\mathbf{u}(t)$ is a function of time, i.e. $\mathbf{u} = \mathbf{u}(t)$. This is known as the **open-loop** optimal control problem. The optimal control is a function of time.

10.1.1 The optimal control Formulation

Start with nonlinear dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n, \quad t \in [t_0, t_f]$$

We write a cost functional expressing our control goals

$$\mathcal{J}(\mathbf{x}, \mathbf{u}) := \Phi(\mathbf{x}(t_f), t_j) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t$$

by means of a running cost $L(\mathbf{x}(t), \mathbf{u}(t))$ and a terminal penalty $\Phi(\mathbf{x}(t_f), t_f)$. We cast the control synthesis as a nonlinear optimization problem

$$\min_{\mathbf{u}(\cdot)} \mathcal{J}(\mathbf{x}(\cdot), \mathbf{u}(\cdot)), \quad \text{subject to } \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}) \tag{1}$$

10.2 Using Calculus of Variations

Lemma 10.4 (Euler-Lagrange to solve (1))**.** *Formally we adjoin the nonlinear constraints through a time-dependent Lagrange multiplier* $\mathbf{p}(t)$ *to the cost functional, leading to*

$$\bar{\mathcal{J}} := \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L[\mathbf{x}(t), \mathbf{u}(t)] + \mathbf{p}^T(t)[\mathbf{f}(t, \mathbf{x}, \mathbf{u}) - \dot{\mathbf{x}}]\, \mathrm{d}t$$

At this point, we define the ***Hamiltonian*** *as:*

$$\mathcal{H}(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t)) := L(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{p}^T(t)\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t))$$

Integration by parts of the last term ($\mathbf{p}^T\dot{\mathbf{x}}$ *)in* $\bar{\mathcal{J}}$ *yields*

$$\overline{\mathcal{J}} = TO\ FINISH$$

FINISH THIS LATER

10.3 Pontryagin's Maximum Principle

Characterises optimality conditions for a problem of the type

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n, \mathbf{u} \in \mathcal{U} \subset \mathbb{R}^m, \quad t \in [t_0, t_f]$$

With a cost functional, also known as the **Bolza Problem**

$$\mathcal{J}(\mathbf{x}, \mathbf{u}) := \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t))\,\mathrm{d}t$$

and terminal constraints

$$\Psi(\mathbf{x}(t_f)) = 0, \quad \Psi : \mathbb{R}^n \to \mathbb{R}^q$$

Differences: the terminal time t_f is allowed to be free, and we express a set of q terminal constraints for the final state through $\Psi(\mathbf{x}(t_f))$ and the existence of a space of admissible controls $\mathcal{U}$, where we restrict our optimal control signal.
Recalling Hamiltonian

$$\mathcal{H}(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t)) = L(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{p}^T(t)\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t))$$

the optimality conditions now read

The last condition states that the optimal control is the minimizer of the Hamiltonian. This is equivalent ot the previous condition that $\frac{\partial H}{\partial u} = 0$, but since we include constraint $\mathbf{u} \in \mathcal{U}$, we realise this as

$$\mathbf{u}^* \in \arg\min_{\mathbf{w} \in \mathcal{U}} \mathcal{H}(t, \mathbf{x}^*(t), \mathbf{w}, \mathbf{p}^*(t))$$