EXERCISE SHEET WEEK 2

There will be more rejection examples also in the following week's exercises.

Exercise 2.1. Show that if $X \sim$ Exponential$(x; 1)$ then $Y = \alpha X^{1/\beta}$ has the Weibull distribution with density

$$p_Y(y) = \beta\alpha^{-\beta}y^{\beta-1}\text{exp}\left[-\left(\frac{y}{\alpha}\right)^{\beta}\right]$$

Hence, explain how you could generate Weibull random quantities using the inversion method and only uniform random numbers (no use of exponentials). Implement and test this algorithm. Plot your histogram against the Weibull density.

Exercise 2.2. (Sampling non-uniformly on the circle) Recall that we discussed in the course that sampling

$$r \sim \text{Unif}(0, 1), \quad \theta \sim \text{Unif}((-\pi, \pi)$$

and setting

$$X_1 = r\cos\theta, \quad Y_2 = r\sin\theta$$

will **not** give us uniform sample on the circle (See Example 2.6).

(a) Prove that, this sampler samples from

$$p_{x_1,x_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{x_1^2 + x_2^2}} \qquad \text{for } x_1^2 + x_2^2 \leq 1.$$

(b) Plot this function in 2D on a regular grid and verify that it is not uniform. Compute the integral of this function over the unit circle and verify that it is 1 (analytically or numerically).

(c) Give a reasoning why the samples look the way they look in Figure 2.4 in the notes.

Exercise 2.3. Consider the box example given in the course:

$$p(x) = \text{Beta}(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}.$$

where $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ and choose:

$$q(x) = \text{Unif}(x; 0, 1)$$

Show that, as stated in the lecture, $M^\star = 1.5$.

Exercise 2.4. Define

$$\text{Gamma}(x; \alpha, \theta) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x}\theta^{\alpha}, \quad \text{for } x > 0, \quad \alpha, \theta > 0.$$

Recall the Beta density

$$\text{Beta}(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}.$$

(a) Prove that if $X_1 \sim \text{Gamma}(\alpha, 1)$ and $X_2 \sim \text{Gamma}(\beta, 1)$, then

$$Y = \frac{X_1}{X_1 + X_2} \sim \text{Beta}(\alpha, \beta).$$

(b) Test this as a sampler. You should **only use uniforms** (`rng.uniform`). Precisely:

(i) Remind yourself the code given in the webpage for sampling $\text{Exp}(x; \lambda)$. Convert this into a function `exp_sampler` which uses only uniforms.

(ii) Write a sampler function for `gamma_sampler` to sample $\text{Gamma}(x; \alpha, 1)$ using rejection sampling as in Example 2.12, using optimal q_λ for given α you want to sample from.

(iii) Then set α, β to prespecified values of your choice and build your Beta sampler using the transformation above. Test resulting histogram against the density (use `scipy.stats.beta` to plot the density).

Exercise 2.5 (Exercise 2.8 from *Introducing Monte Carlo Methods with R*, Christian Robert, George Casella)**.** Consider the rejection sampling method for $p(x) = \mathcal{N}(x; 0, 1)$ and choose

$$q_\alpha(x) = (\alpha/2) \exp(-\alpha|x|)$$

as the proposal with $\alpha > 0$.

(a) Show that

$$M_\alpha = \sup_x \frac{p(x)}{q_\alpha(x)} = \sqrt{\frac{2}{\pi}} \alpha^{-1} e^{\alpha^2/2}.$$

Find the optimal α, i.e., the α that minimises M_α:

$$\alpha^\star = \arg\min_\alpha M_\alpha.$$

(b) Show that the acceptance rate $\hat{a}$ as defined during lectures is then $\sqrt{\pi/2e} = .76$.

(c) Empirically show that this is the acceptance rate for this method. You can draw from q_α using `np.random.laplace`. This density is called *Laplace density*. A sample from this density can be taken using

```
x = np.random.laplace(0, 1/alpha),
```

or `rng.laplace(0, 1/alpha)` as a better method when you initialise `rng`. Using this density as a proposal compute the acceptance rate (i.e. the proportion of the number of *accepted samples* vs. the number of total samples you drew from the proposal) for different α's and plot it against α. Verify with $\alpha^\star$, you numerically get .76.