EXERCISES WEEK 3

Exercise 3.1. Sample from a truncated Normal with

$$\bar{p}(x) = \mathcal{N}(x; 0, 1)\mathbf{1}_{x \in [-0.8, 0.8]}(x)$$

using $q(x) = \mathcal{N}(x; 0, 1)$. Plot the histogram and the unnormalised density. Can you plot the pdf? You can use `np.random.normal` for sampling from $q(x)$.

Exercise 3.2. Simulate data from the nonlinear model

$$p(x) = \text{Unif}(x; -10, 10)$$
$$p(y|x) = \mathcal{N}(y; a\cos(x) + b, \sigma^2),$$

with $a = 0.5$, $b = 0.5$, and $\sigma = 0.15$. You can use `np.random.normal` for this. Scatter plot (x, y) samples and discuss the behaviour. Plot $p(y)$ using samples/histogram, do you think $p(y)$ is computable any other way?

Exercise 3.3. Sample the following mixture of Gaussians using only uniforms:

$$p(x) = \sum_{k=1}^{5} w_k q_k(x),$$

where

$$q_k(x) = \mathcal{N}(x; \mu_k, \sigma_k^2).$$

Use the Box-Müller transform and a transformation to sample Gaussians with a certain mean and covariance – and inversion to sample from the discrete distribution. The weights are defined as

$$w_1 = 0.1,\ w_2 = 0.2,\ w_3 = 0.3,\ w_4 = 0.2,\ w_5 = 0.2,$$

and

$$\mu_1 = -2,\ \mu_2 = -1,\ \mu_3 = 0,\ \mu_4 = 1,\ \mu_5 = 2$$

and

$$\sigma_1 = 0.5,\ \sigma_2 = 0.1,\ \sigma_3 = 0.5,\ \sigma_4 = 0.2,\ \sigma_5 = 0.5.$$

Note that these are standard deviations, not variances. What would you do if each Gaussian was truncated around their mean $[\mu_k - 0.1, \mu_k + 0.1]$? Think about combining your code with a rejection sampler, if mixture components themselves are hard to sample from.

Exercise 3.4 (Curse of dimensionality for rejection samplers)**.** Assume $x \in \mathbb{R}^d$ and $p(x)$ and $q(x)$ are two d-dimensional probability distributions. If

$$p(x) = \prod_{i=1}^{d} p_0(x_i) \quad \text{and} \quad q(x) = \prod_{i=1}^{d} q_0(x_i)$$

where $x_i \in \mathbb{R}$ and

$$\sup_{x \in \mathbb{R}} \frac{p_0(x)}{q_0(x)} = K,$$

then find M in terms of K and prove that the acceptance rate in rejection sampling will go to zero as $d \to \infty$.

Apply this result to

$$p(x) = \mathcal{N}(x; 0, \sigma_p^2 I_d)$$
$$q(x) = \mathcal{N}(x; 0, \sigma_q^2 I_d),$$

where $\sigma_q > \sigma_p$. Find the optimal M and show that the acceptance rate will approach zero as $d \to \infty$.

Exercise 3.5. This question is simple, but requires you to remember the discussion during the lecture. Recall that we discussed how to sample uniformly from a circle. Let the uniform distribution on the circle be

$$p_{x_1,x_2}(x_1, x_2) = \frac{1}{\pi}, \qquad x_1^2 + x_2^2 \leq 1.$$

(i) Compute the marginals in x_1, x_2, i.e., write the expressions of $p_{x_1}(x_1)$ and $p_{x_2}(x_2)$. **Hint:** Recall how you compute marginals from joints and just apply the rule.

(ii) Sample from these marginals and plot the density vs. histograms to prove that your samples are coming from the correct marginals. **Hint:** How do you obtain samples from marginals given the samples from the joint? Do not forget to scatter plot your samples in 2d to ensure that your joint samples are uniform.